# Finding Minimum Witness Sets in Orthogonal Polygons

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# Finding Minimum Witness Sets in Orthogonal Polygons

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## Abstract

A *witness set* $W$ in a polygon $P$ is a subset of $P$ such that any set $G \subset P$ that guards $W$ is guaranteed to guard $P$. We study the problem of finding a minimum witness set for an orthogonal polygon under three models of orthogonal visibility: rectangular, staircase and $k$-periscope visibility.

Under the traditional line-segment visibility, it is known that not all simple polygons admit a finite witness set and, when a polygon admits a finite minimal witness set, the witnesses must lie on the boundary of the polygon [3].

In this paper, we prove that every orthogonal polygon with $n$ vertices admits a finite witness set which has $O(n^2)$ witnesses under rectangular, staircase and $k$-periscope visibility. We also show that there exist orthogonal polygons which require $\Omega(n^2)$ witnesses under staircase visibility. Furthermore, we show that there exist orthogonal polygons for which the boundary is not a witness set for any of the three considered visibility models. Finally, we describe an $O(n^4)$ time algorithm to find a minimum witness set for a given orthogonal polygon under the rectangular and staircase visibility models.

## 1 Introduction

The Art Gallery Problem (AGP) is a classical problem in Computational Geometry that has been widely studied since it was proposed in 1973 by V. Klee [2]: given a polygon $P$, the Art Gallery Problem consists in finding a minimum set of points $G$ such that each point in $P$ is guarded by at least one element of $G$. $G$ is called a *guard set*.

Several variants of AGP arise by imposing restrictions on the type of guards, the visibility model or the shape of the gallery. Many results on problems related to AGP can be found in the book by O'Rourke [9] and the surveys by Shermer [12] and Urrutia [13].

Under line-segment visibility, it is well known that finding minimum guard sets for simple and orthogonal polygons is an NP-hard problem, see Lee and Lin [7] and Schuchardt and Hecker [10], respectively.

In this paper we are interested in three kinds of orthogonal visibility: rectangular, staircase and $k$-periscope visibility. For rectangular visibility, a minimum guard set can be found in polynomial time if the polygon has no holes. An algorithm with complexity $O(n^{17})$ is given by Worman and Keil in [14]. However, if the polygon has holes, then AGP is NP-hard under rectangular visibility, as proven by Biedl and Mehrabi in [1]. For staircase visibility, Motwani et al. [8] prove that a minimum guard set can be found in $O(n^8)$ time in orthogonal polygons without holes. It remains as an open problem to determine if AGP is NP-hard in polygons with holes under staircase visibility. Finally, for $k$-periscope visibility, Gewali and Ntafos proved in [6] that AGP can be solved in $O(n^3)$ time for $k = 1$ in a restricted class of orthogonal polygons.

The *Witness Problem* is a variant of AGP that consists in finding a set $W$ in a given polygon, such that if $W$ is guarded by a set of guards $G$, then the polygon is guaranteed to be guarded by $G$. The set $W$ is called a *witness set*. A motivation behind this research is that a witness set allows us to quickly verify if a polygon is guarded by a set of points.

The Witness Problem under line-segment visibility in simple polygons was studied by Chwa et al. in [3]. They proved that not all simple polygons admit a finite witness set. They also proved that, if a simple polygon $P$ admits a finite minimal witness set, then all the witnesses must lie on the boundary of $P$. In addition, they gave an $O(n^2 \log n)$ time algorithm that computes a minimum witness set for $P$ if it exists, or it reports the non-existence of a finite witness set otherwise.

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In this paper, we study the Witness Problem in orthogonal polygons under rectangular, staircase and $k$-periscope visibility. First, we show that there exist orthogonal polygons such that their boundary is not a witness set under the visibility models considered here. Then, we prove that we can always find a finite witness set with $O(n^2)$ elements in any orthogonal polygon under each of the considered visibility models. Finally, we describe an $O(n^4)$ time algorithm for finding a minimum witness set in orthogonal polygons under rectangular and staircase visibility models.

## 2 Preliminaries

Let $P$ be a simple polygon. A vertex of $P$ is *convex* if its interior angle is less than $\pi$ and is *reflex* if its interior angle is greater than $\pi$.

Consider two points $p$ and $q$ in $P$. Under the *line-segment* visibility model, $p$ and $q$ are mutually visible if the line segment $\overline{pq}$ is contained in $P$. Under the *rectangular* visibility model, $p$ and $q$ are mutually visible if the smallest isothetic rectangle containing $p$ and $q$, denoted by $R(p, q)$, is contained in $P$. Under the *staircase* visibility model, $p$ and $q$ are mutually visible if $P$ contains a monotone isothetic polygonal path joining $p$ and $q$. Finally, under the *$k$-periscope* visibility model, for $k \in \mathbb{N} \setminus \{0\}$, $p$ and $q$ are mutually visible if $P$ contains an isothetic polygonal path joining $p$ and $q$ with at most $k$ bends.

The following definitions are common to all the visibility models described above. The *kernel* of $P$, denoted by $\mathrm{K}(P)$, is the set of points in $P$ from which every point in $P$ is visible. Let $p$ be a point in $P$. The *visibility polygon* of $p$, denoted by $\mathrm{VP}(p)$, is the set of points of $P$ that are visible to $p$. The *visibility kernel* of $\mathrm{VP}(p)$, for short the *visibility kernel of* $p$, denoted by $\mathrm{VK}(p)$, is the set of points from which each point of $\mathrm{VP}(p)$ is visible.

Recall that a witness set is defined as a set of points in a given polygon $P$, such that if any set of points $G$ guards $W$, then $P$ is also guarded by $G$. We say that a point $p$ *witnesses* another point $q$ if guarding $p$ guarantees that $q$ is also guarded.

The following auxiliary results were proved for line-segment visibility. Nevertheless, they also hold for rectangular, staircase and $k$-periscope visibility models. This is because they rely on properties of visibility polygons that are valid in all the visibility models mentioned above.

**Theorem 1** *[3, Theorem 1] Let $P$ be a simple polygon and let $W$ be a point set in $P$. Then $W$ is a witness set for $P$ if and only if the union of the visibility kernels of the elements of $W$ covers $P$ completely.*

**Lemma 2** *[3, Lemma 1] Let $P$ be a polygon and let $p$ and $q$ be points in $P$. Then $p$ witnesses $q$ if and only if $q$ lies in $\mathrm{VK}(p)$.*

**Lemma 3** *[3, Lemma 2] Let $P$ be a simple polygon. A point $p$ in $P$ witnesses a point $q$ in $P$ if and only if $\mathrm{VP}(p) \subset \mathrm{VP}(q)$.*

**Lemma 4** *[3, Lemma 3] Let $P$ be a simple polygon, and let $p$, $q$ and $r$ be points in $P$. If $p$ witnesses $q$ and $q$ witnesses $r$, then $p$ witnesses $r$.*

We now give two definitions of directed graphs that we use in the next section. Two nodes $u$ and $v$ in a directed graph are said to be mutually adjacent if there is an arc from $u$ to $v$ and there is an arc from $v$ to $u$. A *clique* in a directed graph is a set of pairwise mutually adjacent nodes of the directed graph.

The *pixelation* of $P$ is the partition of $P$ obtained by extending a horizontal and a vertical line inward at every reflex vertex until each line hits the boundary. The regions obtained from this partition are known as *pixels*. We denote as $\Psi$ the set of pixels obtained from the pixelation of an orthogonal polygon $P$. Note that, in general, $\Psi$ may have a quadratic amount of elements.

## 3 Witnessing orthogonal polygons

It is known that, under line-segment visibility, if there exists a finite witness set $W$, then the elements of a minimal witness set $W$ are always be placed on the boundary of the polygon [3]. For orthogonal polygons under rectangular, staircase or $k$-periscope visibility that is not always the case, even though we can always find a finite witness set for an orthogonal polygon under these three visibility models (as proven below in Lemma 7). In Figure 1 we show an orthogonal polygon that is not witnessed even if we place a witness at each point of its boundary for each of the considered visibility models.

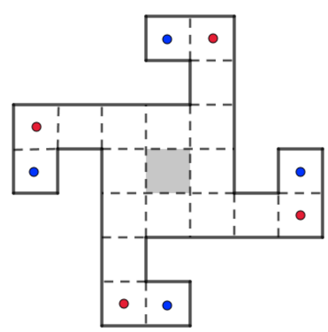

Figure 1: An orthogonal polygon $P$ such that its boundary is not a witness set. The red points guard the boundary of $P$ under rectangular visibility. The blue points guard the boundary of $P$ under 1-periscope and staircase visibility. In both cases the gray region remains unguarded. Hence, there has to be a witness in the interior of $P$. To attain the same effect for $k$-periscope visibility we only need to bend $k-1$ times each extremity of $P$.

Now, we prove that it is always possible to find a finite witness set for an orthogonal polygon under rectangular, staircase and $k$-periscope visibility. Consider the set of pixels $\Psi$ obtained from the pixelation of an orthogonal polygon $P$. We next show that any two points in a pixel of $\Psi$ have the same visibility kernel.

**Lemma 5** *Let $X$ be a pixel obtained from the pixelation of $P$ and let $p, q \in X$ be two distinct points. Then* $\mathrm{VK}(p) = \mathrm{VK}(q)$*, and $X \subset \mathrm{VK}(p)$ under rectangular, staircase and $k$-periscope visibility.*

**Proof.** Let $H$ be the maximal rectangle contained in $P$ whose top edge contains the top edge of $X$ and whose bottom edge contains the bottom edge of $X$. Similarly, let $V$ be the maximal rectangle contained in $P$ whose left edge contains the left edge of $X$ and whose right edge contains the right edge of $X$. Note that the left and right edges of $H$ and the top and bottom edges of $V$ are contained in edges of $P$.

Note that if $\mathrm{VP}(p) = \mathrm{VP}(q)$, then $\mathrm{VK}(p) = \mathrm{VK}(q)$. Therefore we prove that $\mathrm{VP}(p) = \mathrm{VP}(q)$. Consider a point $r \in P$ seen by $p$. Now we prove that $r$ is also visible to $q$. Note that if $r$ is contained in $H$ or $V$ then $r$ is trivially visible from $q$ under any of the three visibility models. Therefore, we suppose that $r$ is not in $H$ nor $V$.

First, we consider rectangular visibility, see Figure 2a. As $r$ is seen by $p$, $R(p, r)$ is contained in $P$. Observe that the horizontal edges of $R(p, r)$ and $R(q, r)$ incident to $p$ and $q$, respectively, are contained in $H$. Similarly, the vertical edges of $R(p, r)$ and $R(q, r)$ incident to $p$ and $q$, respectively, are contained in $V$. As the symmetric difference of $R(p, r)$ and $R(q, r)$ is contained in $V \cup H$ for any $r \in P$ visible to $p$, $R(q, r)$ is contained in $P$. Hence, $q$ sees $r$.

Now consider staircase visibility, see Figure 2b. Since $p$ sees $r$, $P$ contains a monotone isothetic polygonal path $T = t_0, t_1, \ldots, t_k$ joining $p$ and $r$. Let $t_i$ be the line segment of $T$ with an endpoint in $H \cup V$ and the other one outside, and let $\ell$ be the straight line containing $t_i$. Let $s_1$ be the line segment orthogonal to $\ell$ joining $q$ and $\ell$ and let $s = s_1 \cap \ell$. Let $s_2$ be the line segment joining $s$ and $t_i \cap t_{i+1}$. Note that either $t_i$ contains $s_2$ or $s_2$ contains $t_i$. In any case, the isothetic polygonal path $s_1, s_2, t_{i+1}, t_{i+2}, \ldots, t_k$ is monotone and is contained in $P$. Hence, $q$ sees $r$.

Finally, consider $k$-periscope visibility, see Figure 2c. Since $p$ sees $r$, $P$ contains an isothetic polygonal path $T = t_0, t_1, \ldots, t_k$ joining $p$ and $r$ with at most $k$ bends. Let $t_i$ be the first line segment of $T$ with an endpoint in $X$ and the other one outside of $X$, and let $t' = t_{i+1} \cap t_{i+2}$. Let $s$ be the intersection point of the line through $t_{i+1}$ and the line through $q$ parallel to $t_i$. Note that $s$ is either contained in $H$ or $V$. Thus, the polygonal path $\overline{qs}, \overline{st'}, t_{i+2}, t_{i+3}, \ldots, t_k$ joining $q$ and $r$ is completely contained in $P$ and has at most $k$ bends. Hence, $q$ sees $r$.

As $\mathrm{VP}(p)$ for any $p \in X$ is contained in the visibility polygon of any other point in $X$, $X \subset \mathrm{VK}(p)$. $\square$

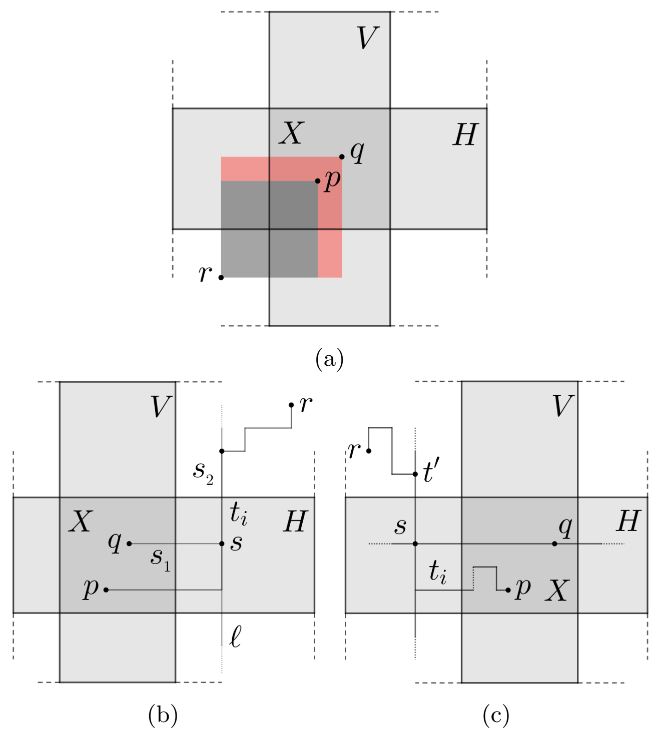


Figure 2: Illustration of the proof for Lemma 5. Given two points $p$ and $q$ in the same pixel $X$, they have the same visibility polygon under: (a) rectangular visibility, (b) staircase visibility and (c) $k$-periscope visibility.

The following corollary is a direct consequence of the previous lemma.

**Corollary 6** *Let $p$ be a point in an orthogonal polygon $P$. Then,* $\mathrm{VK}(p)$ *is the union of a set of pixels of the pixelation of $P$ under rectangular, staircase and $k$-periscope visibility.*

Lemma 5 allows us to give the following definitions. We define the *visibility kernel* of a pixel $a$, denoted as $\mathrm{VK}(a)$, as the visibility kernel of any point in $a$. We say that a pixel $a$ witnesses a pixel $b$ if any point in $a$ contains any point in $b$ in its visibility kernel. Note that, by Lemma 4, if the pixel $a$ witnesses the pixel $b$, then $a$ witnesses the region of $P$ witnessed by $b$.

**Lemma 7** *Let $P$ be an orthogonal polygon with $n$ vertices. There is always a finite witness set $W$ for $P$ under rectangular, staircase and $k$-periscope visibility. Furthermore, $W$ has $O(n^2)$ elements.*

**Proof.** Let $\Psi$ be the set of pixels obtained from the pixelation of $P$. By Lemma 5, the visibility kernel of any point $p$ in $P$ contains the pixel of $P$ containing it. By Theorem 1, a subset $W$ of $P$ is a witness set if the union of the visibility kernels of the elements of $W$ is $P$. Therefore, a set of points containing a point in each pixel of $\Psi$ is a witness set for $P$. Since $|\Psi| \in O(n^2)$, we can always find a finite witness set with $O(n^2)$ elements for an orthogonal polygon under the considered visibility models. □

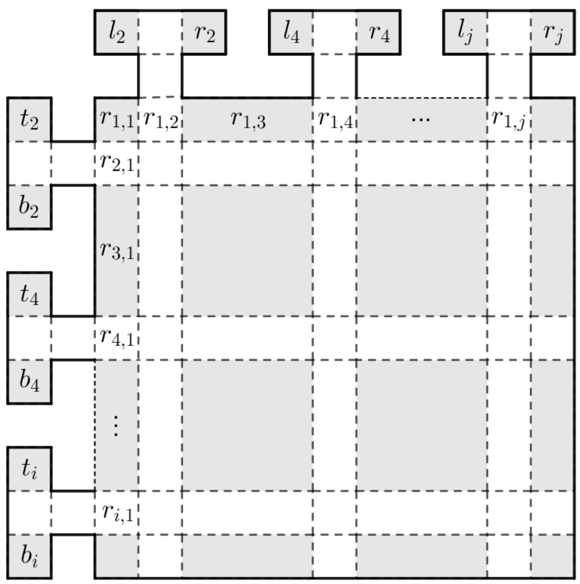


Figure 3: A family of orthogonal polygons that needs a quadratic number of witnesses under staircase visibility.

**Theorem 8** *There are orthogonal polygons with $n$ vertices for which any witness set has cardinality $\Omega(n^2)$ under staircase visibility.*

**Proof.** Let $P$ be an orthogonal polygon consisting of a rectangle $R$ with $m$ vertically oriented $T$-shaped orthogonal polygons attached to the interior of its left edge and $m$ horizontally oriented $T$-shaped orthogonal polygons attached to the interior of its top edge. We illustrate this construction in Figure 3.

Consider the pixelation of $P$. Observe that $R$ is subdivided in a grid with $i$ rows and $j$ columns of pixels, with $i = j = 2m + 1$. We denote as $r_{i,j}$ the pixel at the $i$-th row and the $j$-th column of $R$.

Consider the pixels of the $T$-shaped subpolygons which are shown shaded in Figure 3. We label these pixels as follows. If $T$ is attached to the left edge of $R$ at the $i$-th row of the pixelation we label the top pixel of $T$ as $t_i$ and the bottom pixel of $T$ as $b_i$. If $T$ is attached to the top edge of $R$ at the $j$-th column of the pixelation we label the left pixel of $T$ as $l_j$ and the right pixel of $T$ as $r_j$.

Consider a pixel $r_{i,j}$ with both $i$ and $j$ odd, shown in gray in Figure 3. Then, the set consisting of a guard in each pixel $b_k$ for $k < i$, $t_k$ for $k > i$, $r_k$ for $k < j$ and $l_k$ for $k > j$ guards each pixel in $P$ except $r_{i,j}$. Therefore, we need to place a witness in each of the $r_{i,j}$ with both $i$ and $j$ odd. Note that there are $(m + 1)^2$ such pixels in $R$.

Since $P$ has $n = 16m + 4$ vertices and there are $(m + 1)^2$ pixels in $R$ in which we need to place a witness, it follows that $P$ needs $\Omega(n^2)$ witnesses under staircase visibility. □

It follows from Lemmas 5 and 7 that any minimal witness set contains at most one point in each pixel of $P$. For the sake of simplicity, we will henceforth say that a set of pixels $L$ is a witness set if a set containing a point in each pixel of $L$ is a witness set.

The following remarks follow from Theorem 1 and Lemmas 2, 3 and 4:

- If a pixel $a$ is not contained in the visibility kernel of any other pixel in $P$, then $a$ must be included in the witness set $W$.
- If a pixel $a$ is contained in the visibility kernel of the pixel $b$ but $b$ is not contained in the visibility kernel of $a$, then $a$ cannot be included in a minimum witness set.
- If two or more different pixels contain each other on their respective visibility kernels, then only one of them can be included in a minimal witness set.

### 3.1 An algorithm for finding a minimum witness set

In order to find the pixels contained in a minimum witness set, we first obtain the set of pixels $\Psi$ from the pixelation of $P$. Then, we construct a directed graph $H$, which we call the *kernel graph* of $P$, in such a way that there is a bijection between the set of nodes of $H$ and $\Psi$. After that, we compute the visibility kernel $K$ of the pixel represented by each node $u \in H$. Finally, we add to $H$ the arc from $u$ to $v$ if $K$ contains the pixel represented by $v \in H$, with $u \neq v$. For the sake of simplicity, we say that a node $u$ in $H$ witnesses another node $v$ if $H$ contains the arc $(u, v)$.

To compute the visibility kernel of a point $p$ in an orthogonal polygon under rectangular and staircase visibility, we first compute $\mathrm{VP}(p)$ and then we compute $\mathrm{K}(\mathrm{VP}(p))$, the kernel of $\mathrm{VP}(p)$. It is straightforward to see that, under rectangular visibility, the visibility region $\mathrm{VP}(p)$ of a point is also an orthogonal polygon. For orthogonal polygons without holes, and under staircase visibility, Gewaly [5] proved that the visibility region of a point is also an orthogonal polygon. Therefore, we can use one of the existing algorithms for computing the kernel of an orthogonal polygon under rectangular or staircase visibility.

Now we prove that the polygon obtained in this manner is indeed the visibility kernel of the point $p$. In orthogonal polygons with holes under staircase visibility that is not always the case as shown in Figure 4.

**Proposition 9** *Let $P$ be an orthogonal polygon (possibly with holes). Let $p$ be a point in $P$. Then the polygon obtained by computing the kernel of* $\mathrm{VP}(p)$ *is equal to* $\mathrm{VK}(p)$ *under rectangular visibility.*

**Proof.** It is straightforward to see that $\mathrm{K}(\mathrm{VP}(p))$ is contained in $\mathrm{VK}(p)$. Therefore we only prove that $\mathrm{K}(\mathrm{VP}(p))$ contains $\mathrm{VK}(p)$. Suppose that there exists a point $q \in \mathrm{VK}(p)$ which is not contained in the polygon obtained by computing the kernel of $\mathrm{VP}(p)$. Thus, there exists a point $r \in \mathrm{VP}(p)$ such that $R(q,r)$ is contained in $P$ but not in $\mathrm{VP}(p)$. As $p$ sees both $q$ and $r$, then both $R(p,q)$ and $R(p,r)$ are contained in $P$. Therefore, $R(p,q) \cup R(p,r) \cup R(q,r) \subset P$. This implies that for any point $l \in R(q,r)$ we have that $R(p,l) \subset R(p,q) \cup R(p,r) \cup R(q,r) \subset P$. Therefore, $l \in \mathrm{VP}(p)$, which implies that $R(q,r) \subset \mathrm{VP}(p)$, a contradiction. Hence, $q$ is contained in the polygon obtained by computing the kernel of $\mathrm{VP}(p)$. □

**Proposition 10** *Let $P$ be an orthogonal polygon without holes. Let $p$ be a point in $P$. Then the polygon obtained by computing the kernel of* $\mathrm{VP}(p)$ *is equal to* $\mathrm{VK}(p)$ *under staircase visibility.*

**Proof.** It is straightforward to see that $\mathrm{K}(\mathrm{VP}(p))$ is contained in $\mathrm{VK}(p)$. Therefore we only prove that $\mathrm{K}(\mathrm{VP}(p))$ contains $\mathrm{VK}(p)$. Suppose there exists a point $q \in \mathrm{VK}(p)$ which is not contained in the polygon obtained by computing the kernel of $\mathrm{VP}(p)$. Thus, there exists a point $r \in \mathrm{VP}(p)$ such that $T$, the monotone isothetic polygonal path joining $q$ and $r$, is contained in $P$ but not in $\mathrm{VP}(p)$. As $p$ sees both $q$ and $r$, there exist two monotone isothetic polygonal paths contained in $P$, the first one $T'$ joining $p$ and $q$, and the second one $T''$ joining $p$ and $r$. Since $P$ has no holes, the region $R$ bounded by $T$, $T'$ and $T''$ is contained in $P$. Thus, we can always find a monotone isothetic polygonal path $M$ joining $p$ to any point of $T$, such that $M$ is contained in $R$. Therefore, $p$ sees every point in $T$ which implies that $T$ is contained in $\mathrm{VP}(p)$, a contradiction. Hence, $q$ is contained in the polygon obtained by computing the kernel of $\mathrm{VP}(p)$. □

Now we show how to find a minimum witness set once we have constructed the kernel graph $H$ of $P$. Observe that, by Lemma 4, for any clique $C$ in $H$, any node of $C$ witnesses all the elements of $C$.

We say that a node $u$ of $H$ is a *source node* if for each arc of the form $(v,u)$ for any other node $v$, $H$ contains the arc $(u,v)$.

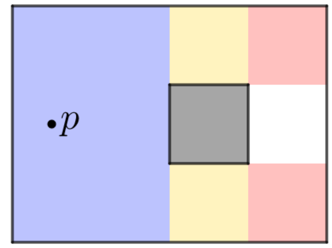


Figure 4: An orthogonal polygon with a hole (shown in gray) and an interior point $p$. Under staircase or 1-periscope visibility the following holds. The visibility polygon of $p$ is the union of the blue, yellow and red regions. The visibility kernel of $p$ is the union of the blue and red regions. The kernel of the visibility polygon of $p$ is the blue region.

**Theorem 11** *Let $P$ be an orthogonal polygon. Let $H$ be the kernel graph of $P$. Let $C$ be a set containing one node in each maximal clique of source nodes in $H$. Then any set containing exactly one point in the pixel represented by each node of $C$ is a minimum witness set for $P$.*

**Proof.** Let $u$ be a node of $H$. If $u$ is a source node then it can only be witnessed by a node contained in a clique containing $u$. Note that, since witnessing is transitive (Lemma 4), each node in $H$ is contained in at most one maximal clique. Therefore, we need one witness per maximal clique of source nodes, placed in any of the pixels associated to the nodes of the clique.

Now suppose that $u$ is not a source node. As witnessing is transitive, there exists an arc from a source node to $u$. Otherwise, $u$ would be a source node. Therefore, it is not necessary to place a witness in a pixel corresponding to a non-source node in $H$.

Hence, the witness set composed by a pixel for each maximal clique of source nodes in $H$ is a minimum witness set for $P$. □

In order to report a minimum witness set, we do a traversal of $H$ as follows. If the node $u \in H$ is not a source node, we remove it from $H$. If $u$ is a source node, we add the pixel represented by $u$ to the witness set $W$ and remove $u$ as well as its neighborhood from $H$. Note that in this manner we add to the witness set at most one pixel for each maximal clique of source nodes in $H$.

Now we analyze the running time of the proposed solution. Obtaining the pixelation takes $O(n^2)$ time, since we need to report $O(n^2)$ regions. The time required to create the directed graph $H$ depends on the subroutines used to compute the visibility kernel of each pixel.

In their book [4], Fink and Wood give an $O(n \log n)$ time algorithm to obtain $\mathrm{VP}(p)$ from a point $p$ in an orthogonal polygon under rectangular visibility. In [11], Schuierer and Wood give an $O(n)$ time algorithm to obtain the kernel of an orthogonal polygon under rectangular visibility. Note that, by Corollary 6, VK($p$) is a set of pixels under rectangular visibility. Note also that VK($p$) is a rectangle. Therefore, finding the coordinates enclosing the set of pixels in VK($p$) takes $O(1)$ time. Since the visibility kernel of a point may have $O(n^2)$ pixels, $H$ may have $O(n^4)$ arcs. Therefore, constructing $H$ takes $O(n^4)$ time under rectangular visibility.

In [5], Gewali gives an $O(n)$ time algorithm to obtain VP($p$) from a point $p$ in an orthogonal polygon without holes under staircase visibility. In the same paper, he gives an algorithm to obtain the kernel of a point in $O(n)$ time for orthogonal polygons without holes under staircase visibility. Once we have computed the visibility kernel of a point, it is not difficult to see that we can find all the pixels it contains in $O(n^2)$ time. Since we only do this once for each pixel, this step takes $O(n^4)$ time. Therefore, constructing $H$ takes $O(n^4)$ time under staircase visibility.

In the last step of the algorithm we do a traversal of $H$ to report the obtained minimum witness set. Since we process each node of $H$ just once, this step takes $O(n^2)$ time. Therefore, our procedure takes $O(n^4)$ overall time under rectangular and staircase visibility. For the case of $k$-periscope visibility, we can achieve the same running time under the assumption that we can obtain the visibility kernel of a point in $O(n^2)$ time. However, efficiently calculating the visibility kernel under $k$-periscope visibility is, to the best of our knowledge, an open problem.

## 4 Conclusions

In this paper we studied the Witness Problem on orthogonal polygons under three models of orthogonal visibility. We proved that there are orthogonal polygons that are not witnessed by their boundary under rectangular, staircase and $k$-periscope visibility. Next proved that all orthogonal polygons admit a finite witness set under these three visibility models. We achieved this by using the so called pixelation of an orthogonal polygon, in which any two points in the same pixel turned out to have the same visibility polygon. We also proved that, under staircase visibility, some orthogonal polygons require a quadratic number of witnesses. As the main result, we gave an $O(n^4)$ time algorithm for computing a minimum witness set for orthogonal polygons under the rectangular and staircase visibility models. This algorithm makes use of the pixelation of a polygon, and relies on an algorithm for computing the visibility kernel of a point under each visibility model.

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