Solution Of Inverse Problem Cauchy Type (Design) For Plane Layer

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Abstract. In engineering practice not always is possible the measurements of temperature on both side of wall (for example turbine casing or combustion chamber). On the other hand it is possible measurement both temperature and heat flux on outside wall. For transient heat conduction equation the measurements temperature and heat flux supplemented by initial condition states Cauchy problem which is ill conditioned. In paper the stable solution is obtained for Cauchy problem by using Laplace transformation and minimisation of continuity in process of integration of convolution. Test examples confirm proposed algorithm of solution of inverse problem.

Keywords: Inverse heat conduction · Cauchy problem · Laplace transformation · Regularization.

1 Introduction

In thermal problems, the coefficients of governing equations such as the thermal conductivity, density and specific heat, as well as the intensity and location of internal heat sources, if they exist, and appropriate boundary and initial conditions should be specified. Such problems are referred to as ‘direct thermal problems’ and may be accurately solved by standard numerical meth-ods since they are well posed. However, in many practical applications which arise in engineering, a part of the boundary is not accessible for heat flux or temperature measurements. For example, the temperature or the heat flux may be serious affected by the presence of a sensor and, hence, there is a loss of accuracy in measurement or, more simply, the surface of the body may be unsuitable for attaching a sensor to measure the temperature or the heat flux. As examples can be inner surface of turbine casing or combustion chamber. The situation when neither the heat flux nor temperature can be determined on a portion of the boundary, while both of them are prescribed on the remaining portion, leads to an ill-posed problem termed the ‘Cauchy problem’. The Cauchy problem is an ill-posed problem and it is more difficult to solve both analytically and numerically.

The Cauchy problem is not new in literature [1–7]. Due to its ill-posed character many approximate method was used. In paper [1] problem is reduced to a linear integral Volterra equation of II type which admits a unique solution. Method of fundamental solution was used in paper [2] for solution steady Cauchy problem. In papers [3, 4] method of finite deference with Fourier transform techniques was used. Legandre polynomials was used in paper [5] for solution 1-D Cauchy problem. Wavelet-Galerkin method with Fourier transform was used in paper [6]. The unique of solution of Cauchy problem was considered in paper [7].

The purpose of this paper is proposition of the stable solution is obtained for Cauchy problem by using Laplace transformation and minimisation of continuity in process of integration of splice. Test examples confirm proposed algorithm of solution of inverse problem.

2 Fundamental Equation

For region shown on Fig. 1 the governing equation and conditions describing heat flow are the following:

- heat conduction equation:

$$\rho c \cdot \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right), x \in (0, \delta), t > 0 \tag{1}$$

- initial condition:

$$T(x, 0) = T_0(x) \tag{2}$$

- boundary conditions:

$$T(x = \delta, t) = H(t) \tag{3}$$

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=\delta} = Q(t) \tag{4}$$

$$T(x = 0, t) = F(t) \tag{5}$$

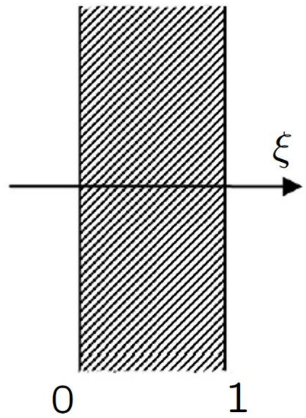


Fig. 1. Calculation area

In consideration of the boundary conditions [3] and [4] the problem formulated by [1–4] is Cauchy problem. For the next considerations the following non-dimensional variables are introduced:

$$T_{max} = \max_{x\in\langle 0,1\rangle, t\geq 0} \left(T(x,t)\right), \quad \vartheta = \frac{T}{T_{max}}, \quad \xi = \frac{x}{\delta}, \quad \tau = \frac{\lambda}{\rho c}\cdot\frac{t}{\delta^2} \tag{6}$$

and now non-dimensional formulation of problem is the following:

- heat conduction equation

$$\frac{\partial\vartheta}{\partial\tau} = \frac{\partial^2\vartheta}{\partial\xi^2}, \quad \xi\in(0,1), \quad \tau>0 \tag{7}$$

- initial condition

$$\vartheta(\xi,0) = \vartheta_0(\xi), \quad \xi \in < 0,1 > \tag{8}$$

- boundary condition at surface $\xi = 1$

$$\vartheta(1,\tau) = h(\tau), \quad h = H\cdot T_{max}, \quad \tau>0 \tag{9}$$

$$-\frac{\partial\vartheta(1,\tau)}{\partial\xi} = q(\tau), \quad q = \frac{\delta}{\lambda\cdot T_{max}}\cdot Q, \quad \tau>0 \tag{10}$$

- unknown boundary condition at surface $\xi = 0$

$$\vartheta(0,\tau) = \chi(\tau), \quad \tau>0 \tag{11}$$

In consideration of linearity of equations (6-9) for their solution will be used Laplace transformation. Let

$$\mathcal{L}\vartheta(\xi,\tau) = \bar{\vartheta}(\xi,s) = \int_0^\infty \vartheta(\xi,\tau)\cdot e^{-s\tau}d\tau \tag{12}$$

then system of equations (6-9) is transformed to form:

- heat conduction equation with initial condition:

$$\bar{\vartheta}(\xi,s) - s\cdot\vartheta(\xi,0) = \frac{d^2\bar{\vartheta}}{d\xi^2}, \quad \vartheta(\xi,0) = \vartheta_0(\xi) \tag{13}$$

- boundary conditions at surface $\xi = 1$:

$$\bar{\vartheta}(1,s) = h(s) \tag{14}$$

$$\frac{-d\bar{\vartheta}(\xi,s)}{d\xi} = \bar{q}(s) \tag{15}$$

- unknown boundary condition at surface $\xi = 0$:

$$\bar{\vartheta}(0,s) = \bar{\chi}(s) \tag{16}$$

Idea of determination of unknown distribution (14) is based on determination of direct problem, namely solution of equation (11) with condition (13) and (14) and in the next determination relation between functions $f(t)$ and $g(t)$.

For simplicity it is assumed $\vartheta_0(\xi) = \vartheta_0 = const$, then solution of direct problem has form:

$$\begin{gathered}\bar{\vartheta}(\xi,s) = \bar{\chi}(s)\cdot\frac{\cosh\sqrt{s}(1-\xi)}{\cosh\sqrt{s}} + \\ +\bar{q}(s)\cdot\frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}} + \frac{\vartheta_0}{s}\cdot\left(1 - \frac{\cos h\sqrt{s}(1-\xi)}{\cosh\sqrt{s}}\right)\end{gathered} \tag{17}$$

For $\xi = 1$:

$$\bar{\vartheta}(1,s) = \bar{\chi}(s)\cdot\frac{1}{\cosh\sqrt{s}} + \bar{q}(s)\frac{\tanh\sqrt{s}}{\sqrt{s}} + \frac{\vartheta_0}{s}\left(1 - \frac{1}{\cosh\sqrt{s}}\right) \tag{18}$$

Unknown function $\bar{\chi}(s)$ we will search on the base known distribution (9), namely

$$\bar{\vartheta}(1,s) = s\cdot\bar{\chi}(s)\cdot\frac{1}{s\cdot\cosh\sqrt{s}} + \bar{q}(s)\frac{\tanh\sqrt{s}}{\sqrt{s}} + \frac{\vartheta_0}{s}\left(1 - \frac{1}{\cosh\sqrt{s}}\right) = \bar{h}(s)$$

In this way we have Volterra integral equation second kind for determination of function $\chi(\tau)$ in form

$$\begin{gathered}\mathcal{L}^{-1}\left[s\cdot\bar{\chi}(s)\right] * \mathcal{L}^{-1}\left[\frac{1}{s\cosh\sqrt{s}}\right] + \mathcal{L}^{-1}\left[s\bar{q}(s)\right] * \mathcal{L}^{-1}\left[\frac{\tanh\sqrt{s}}{s\sqrt{s}}\right] + \\ +\vartheta_0\left\{\eta(\tau) - \mathcal{L}^{-1}\left[\frac{1}{s\cosh\sqrt{s}}\right]\right\} = h(\tau))\end{gathered}$$

Therefore solution in transformation region

$$\begin{gathered}\bar{\vartheta}(\xi,s) = s\cdot\bar{\chi}(s)\cdot\frac{\cosh\sqrt{s}(1-\xi)}{s\cdot\cosh\sqrt{s}} + \\ +s\cdot\bar{q}(s)\cdot\frac{1}{s}\cdot\frac{\sinh\sqrt{s}\xi}{\sqrt{s}\cdot\cosh\sqrt{s}} + \vartheta_0\left(\frac{1}{s} - \frac{1}{s\cdot\cosh\sqrt{s}}\right), \quad \xi\in\langle 0,1\rangle\end{gathered} \tag{19}$$

Poles of transform (19) are given by equations

$$s = 0 \quad and \quad \cosh\sqrt{s} = 0 \tag{20}$$

Therefore putting $\sqrt{s} = i \cdot \mu$ we have equation

$$\cosh\sqrt{s} = \cosh\ i \cdot \mu = \cos\ \mu = 0$$

then

$$\mu_n = (2n-1) \cdot \frac{\pi}{2}, \quad n = 1, 2, ... \tag{21}$$

In this way

$$\begin{gathered}
\mathcal{L}^{-1}\left[\frac{\cosh\sqrt{s}(1-\xi)}{s \cdot \cosh\sqrt{s}}\right] = \\
= \underset{s=0}{res}\frac{\cosh\sqrt{s}(1-\xi)}{s \cdot \cosh\sqrt{s}} + \sum_{n=1}^{\infty} \underset{s=s_n}{res}\frac{\cosh\sqrt{s}(1-\xi)}{s \cdot \cosh\sqrt{s}} \cdot e^{s \cdot \tau} = \\
= 1 + \sum_{n=1}^{\infty} \underset{s=s_n}{lim}\frac{(s-s_n) \cdot \cosh\sqrt{s}(1-\xi)}{s \cdot \cosh\sqrt{s}} \cdot e^{s\tau} = \\
= 1 - 2\sum_{n=1}^{\infty} \frac{2\cosh\mu_n(1-\xi)}{\mu_n \cdot \sin\mu_n} \cdot e^{-\mu_n^2 \cdot \tau} = \\
= 1 - \frac{4}{\mu}\sum_{n=1}^{\infty} \frac{\sin\mu_n\xi}{2n-1} \cdot e^{-\mu_n^2 \cdot \tau} = \\
\mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{\sin\sqrt{s}\xi}{\sqrt{s} \cdot \cos 2}\right] = \xi - 2\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{\sin\mu_n\xi}{\mu_n^2} \cdot e^{-\mu_n^2 \cdot \tau}
\end{gathered} \tag{22}$$

Since

$$\mathcal{L}[q'(\tau)] = s \cdot q(s) - q(0) \quad and \quad \mathcal{L}^{-1}[sq(s)] = q'(\tau) + q_0 \cdot \delta(\tau)$$

accordingly

$$\vartheta(\xi,\tau)=\mathcal{L}^{-1}\left[\bar{\vartheta}(\xi,s)\right]=\vartheta_0\left[1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau}\right]+$$
$$+\mathcal{L}^{-1}\left[s\cdot\chi(s)\right]*\left[1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau}\right]+$$
$$+\mathcal{L}^{-1}\left[sq(s)\right]*\left(\xi-\frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{(2n-1)^2}\cdot e^{-\mu_n^2\cdot\tau}\right)=$$

$$=\vartheta_0\left[1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau}\right]+\left[\chi'(\tau)+\chi_0\cdot\delta(\tau)\right]*\eta(\tau)-$$
$$-1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau}\cdot\int_0^{\tau}\left[\chi'(p)+\chi_0\cdot\delta(p)\right]\cdot e^{-\mu_n^2\cdot p}\cdot dp+$$
$$+\left[q'(\tau)+q_0\cdot\delta(\tau)\right]*\eta(\tau)\cdot\xi-$$
$$-\frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{(2n-1)^2}\cdot e^{-\mu_n^2\cdot\tau}\cdot\int_0^{\tau}\left[q'(p)+q_0\cdot\delta(p)\right]\cdot e^{-\mu_n^2\cdot p}\cdot dp=$$
$$=\vartheta_0\left[1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\cdot e^{-\mu_n^2\cdot\tau}\right]+\chi(\tau)\cdot\left[1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}\right]+$$
$$+\pi\cdot\sum_{n=1}^{\infty}(2n-1)\cdot\sin(2n-1)\frac{\pi}{2}\xi\cdot e^{-\mu_n^2\cdot\tau}\cdot\int_0^{\tau}\chi(p)\cdot e^{-\mu_n^2\cdot p}\cdot dp+$$
$$+\sum_{n=1}^{\infty}\sin(2n-1)\frac{\pi}{2}\xi\cdot e^{-\mu_n^2\cdot\tau}\cdot\int_0^{\tau}q(p)\cdot e^{-\mu_n^2\cdot p}\cdot dp \tag{23}$$

For $\xi>0$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}\xi}{2n-1}=1$$

then square bracket in (23) at function $\chi(\tau)$ is equal zero for $\xi=0$, $\vartheta(0,\tau)=\chi(\tau)$.

The next consideration is carried out for case when $q(\tau) = 0$ and $\vartheta_0 = 0$, at that time the form of solution (23) can be written in the following form

$$\begin{aligned}\vartheta(\xi,\tau) &= \int_0^\tau \chi(p)\cdot 2\sum_{n=1}^{\infty}\mu_n \cdot \sin\mu_n\xi \cdot e^{-\mu_n^2(\tau-p)}\cdot dp = \\ &= \int_0^\tau \chi(p)\cdot\psi(\xi,\tau,p)\cdot dp, \quad \xi\in(0,1\rangle\end{aligned} \tag{24}$$

where

$$\psi(\xi,\tau,p) = 2\sum_{n=1}^{\infty}\mu_n\cdot\sin\mu_n\cdot e^{-\mu_n^2(\tau-p)}$$

From condition (9) on the base (24) we obtained equation for determination function $\chi(\tau)$

$$\vartheta(1,\tau) = \int_0^\tau \chi(p)\cdot\psi(\xi,\tau,p)\cdot dp = h(\tau) \tag{25}$$

or

$$\int_0^\tau \chi(p)\cdot\psi(1,\tau,p)\cdot dp = h(\tau) \tag{25a}$$

The equation is an integral equation of Volterra kind.

Solution of integral equation (25a)

Function $h(t)$ is temperature at the boundary $\xi = 1$ and in practice is known from the measurements, then is given in the following time steps $\tau_k = k\cdot\Delta\tau, \quad k = 0,1,2,...$, then equation (25a) takes form:

$$\int_0^{\tau_k}\chi(p)\cdot\psi(1,\tau_k,p)\cdot dp = h(\tau_k) \quad or \quad \int_0^{\tau_k}\chi(p)\cdot\psi_k(1,p)\cdot dp = h(\tau_k) = h_k \tag{26}$$

Because

$$\int_0^{\tau_k}\chi(p)\cdot\psi_k(\xi,p)\cdot dp = \sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}\chi(p)\cdot\psi_k(\xi,p)\cdot dp = \sum_{j=0}^{k}\chi_j\cdot\psi_{kj}(\xi) \tag{27}$$

consequently equation (26) for succeeding time while takes form:

$$\begin{cases} k=1 & : \quad \chi_0\cdot\psi_{10}+\chi_1\cdot\psi_{11} = h_1 \\ k=2 & : \quad \chi_0\cdot\psi_{20}+\chi_1\cdot\psi_{21}+\chi_2\cdot\psi_{22} = h_2 \\ k=3 & : \quad \chi_0\cdot\psi_{30}+\chi_1\cdot\psi_{31}+\chi_2\cdot\psi_{32}+\chi_3\cdot\psi_{33} = h_3 \\ \multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{cases} \tag{28}$$

then for $\chi_0 = 0$ system of equation (28) has solution

$$\chi_1 = (h_1/\psi_{11}) \quad \chi_k = \left(h_k - \sum_{j=1}^{k-1} \psi_{kj}(\xi = 1) \cdot \varphi_j \right) / \psi_{kk}, \quad k \geq 2 \tag{29}$$

Let determine elements ψ_{kj} matrix $[\psi]$. These elements ψ_{kj} essentially dependent from way of integration of function $\chi(p)$; the simplest way of integration in (27) can be expressed as

$$\int_0^{\tau_k} \chi(p) \cdot \psi_k(\xi, p) \cdot dp = \sum_{j=1}^{k} \int_{\tau_{j-1}}^{\tau_j} \chi(p) \cdot \psi_k(\xi, p) \cdot dp =$$

$$= \sum_{j=1}^{k} \int_{\tau_{j-1}}^{\tau_j} [\Theta \cdot \chi_{j-1} + (1 - \Theta) \cdot \chi_j] \cdot \psi_k(\xi, p) \cdot dp =$$

$$= \sum_{j=1}^{k} [\Theta \cdot \chi_{j-1} \cdot r_j + (1 - \Theta) \cdot \chi_j \cdot r_j] = \Theta \cdot \chi_0 \cdot r_{k1} +$$

$$+ \sum_{j=1}^{k} \chi_j \left[\Theta \cdot r_{kj+1} + (1 - \Theta) \cdot r_{kj}\right] + (1 - \Theta)\chi_k \cdot r_{kk} = \sum_{j=0}^{k} \chi_j \psi_{kj}$$

$$r_{kj} = \int_{\tau_{j-1}}^{\tau_j} \psi_k(\xi, p) \cdot dp \tag{30}$$

$$\chi_{k0} = \Theta \cdot r_{k1}, \quad \chi_{kj} = \Theta \cdot r_{kj+1} + (1 - \Theta) r_{kj}$$
$$j = 1, ..., \; k - 1, \quad \chi_{kk} = (1 - \Theta) r_{kk}, \quad \Theta \in (0, 1)$$

System of equation (28) can be written in matrix form

$$[\psi] \{\chi\} = \{h\}, \quad \dim[\psi] = M \times M, \quad \dim \{h\} = M \tag{31}$$

Since

$$\psi(\xi, \tau, p) = 2 \sum_{n=1}^{\infty} \mu_n \cdot \sin \mu_n \xi \cdot e^{-\mu_n^2 (\tau - p)}$$

then for $\tau_k = k \cdot \Delta\tau$

$$\begin{aligned} r_{kj} &= \int_{\tau_{j-1}}^{\tau_j} \psi(\xi, \tau, p) \cdot dp = 2 \sum_{n=1}^{\infty} \mu_n \cdot \sin \mu_n \xi \cdot \int_{\tau_{j-1}}^{\tau_j} e^{-\mu_n^2 (\tau - p)} \cdot dp = \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin \mu_n \xi}{\mu_n} \cdot \left(e^{-\mu_n^2 (\tau_k - \tau_j)} - e^{-\mu_n^2 (\tau_k - \tau_{j-1})} \right) = \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin \mu_n \xi}{\mu_n} \cdot \left(e^{-\mu_n^2 \Delta\tau (k-j)} - e^{-\mu_n^2 \Delta\tau (k-j+1)} \right) \end{aligned} \tag{32}$$

It must be noted, that $r_{kj} = r_{k+1,j+1}$ then $\psi_{kj} = \psi_{k+1,j+1}$. This property safes time of calculation in the determination the following matrix elements $[\psi]$.

3 Numerical Calculations

In order to test solution of integral equation (26) we will compare numerical solution with analytical solution. Analytical solution equation (7) with initial condition $\vartheta(\xi, 0) = 0$ and the following boundary conditions

$$\vartheta(\xi = 0, \tau) = T_b \cdot \left(1 - e^{-\beta\tau}\right), \quad -\frac{\partial\vartheta(\xi = 1, \tau)}{\partial\xi} = Bi \cdot \vartheta(\xi = 1, \tau) \tag{33}$$

has form

$$\begin{gathered} \vartheta(\xi, \tau) = T_b \cdot \left(1 - \frac{Bi}{Bi + 1} \cdot \xi\right)\left(1 - e^{-\beta\tau}\right) + \\ +2T_b \cdot \beta \cdot e^{-\beta\tau} \cdot \sum_{n=1}^{\infty} w_n(\xi) \cdot \frac{1}{p_n^2 - \beta} - \\ -2T_b \cdot \beta \cdot \sum_{n=1}^{\infty} w_n(\xi) - \frac{1}{p_n^2 - \beta} \cdot e^{-p_n^2\tau}, \\ w_n(\xi) = -\frac{\sin p_n\xi}{p_n} \cdot \left(1 - \frac{Bi}{Bi^2 + Bi + p_n^2}\right) \end{gathered} \tag{34}$$

where numbers p_n are the following roots of equation

$$\tan p_n = -\frac{p_n}{Bi}, \; n = 1, 2, ..., \text{ and for } Bi \to 0, \; p_n = \frac{\pi}{2}(2n - 1)$$

and

$$\lim_{\tau\to\infty} \vartheta(\xi, \tau) = T_b \cdot \left(1 - \frac{Bi}{Bi + 1} \cdot \xi\right)$$

Solution (34) is used for determination of functions $h(t)$ and $q(t)$ given by formulas (9) and (10).

System (28) is numerical unstable and determination solution from (31) for relatively low values M leads to solution system of equation of order $M - 1$. Then regularization is needed. Oscillations of vector $\{\chi\}$ are appeared at the end of region $\langle 0, M \cdot \Delta\tau\rangle$, Fig.2.

Regularization of solution of system of equations (31)

At each segment $\langle\tau_{j-1}, \tau_j\rangle$, $j = 1, ..., M$ function $\chi(\tau)$ in (30) is approximate by constant $\Theta = \Theta_{j-1} + \Theta_j \cdot (1 - \Theta)$, $0 < \Theta < 1$, therefore between segments function $\chi(\tau)$ is not differentiate and appear jump of first derivative.

Leading first parabola $\chi_{j-2,j-1,1}$ through following points (τ_{j-2}, χ_{j-2}), (τ_{j-1}, χ_{j-1}), (τ_j, χ_j) and second parabola $\chi_{j-1,j,j+1}$ through points (τ_{j-1}, χ_{j-1}),

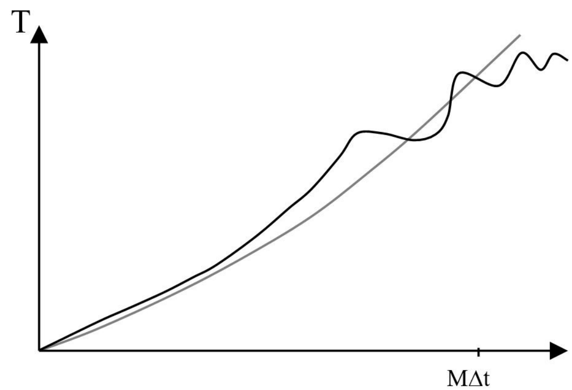


Fig. 2. Oscillations of solution of system of equations (31)

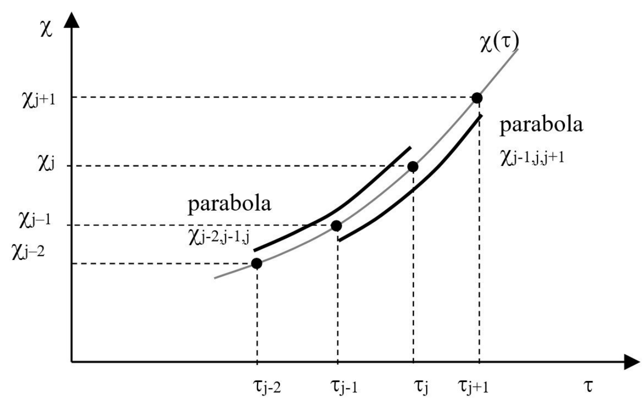


Fig. 3. Idea of parabolic regularization of solution $\chi(\tau)$

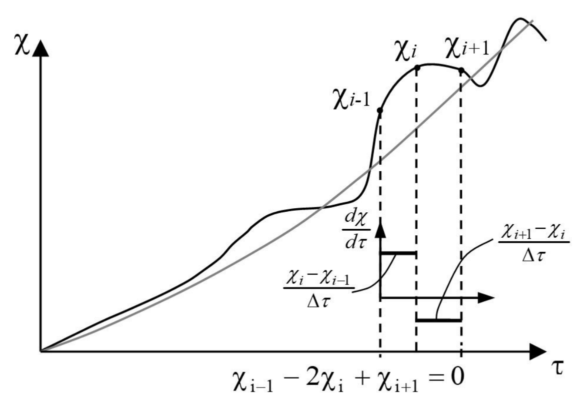


Fig. 4. Idea of linear regularization of solution $\chi(\tau)$

(τ_j, χ_j), (τ_{j+1}, χ_{j+1}) we require that difference between first derivative in common points both parabolas must be equal zero, what in case uniform net τ_{j-1} and τ_j $\tau_k - \tau_{k-1} = h,\ k = 1, 2, ...,\ M$ leads to one equation on region $\langle \tau_{j-2}, \tau_{j+1} \rangle$

$$\chi_{j-2} - 3\chi_{j-1} + 3\chi_j - \chi_{j+1} = 0, \quad j = 2, ..., M-2 \tag{35}$$

In case of linear regularization, as shown in Fig. 4, the conditions that lead to reduction of jump of first derivation are

$$\chi_{j-1} - 2\chi_j + \chi_{j+1} = 0, \quad j = 1, 2, ..., M-1 \tag{36}$$

Dimension of matrix $[\psi]$ is equal $\dim[\psi] = (M+1) \times (M+1)$, whereas for not large values M, $\mathrm{rank}[\psi] = M$, then it is sufficient to add to system of equation (28) condition (35) for $j = M-1$. Let consider more general case, namely added condition (35) or (36) to system (28) with internal knot in number $M-3$. The system of equation is a followin or $M-1$

$$\begin{bmatrix} [\psi] \\ [w] \end{bmatrix} \{\chi\} = \begin{Bmatrix} \{h\} \\ \{0\} \end{Bmatrix}, \quad \dim[\psi] = (M+1) \times (M+1) \tag{37}$$

Where matrix $[w]$ related with condition (35) has form, $I\_Matrix\_Reg = 1$

$$[w] = \begin{bmatrix} 1 & -3 & 3 & -1 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 1 & -3 & 3 & -1 \end{bmatrix}, \ \dim[w] = (M-3) \times (M+1) \tag{38}$$

and for condition (36) we have, $I\_Matrix\_Reg = 0$

$$[w] = \begin{bmatrix} 1 & -2 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 1 & -2 & 1 \end{bmatrix}, \ \dim[w] = (M-1) \times (M+1) \tag{39}$$

Solution of over-determined system of equations (37) can be considered as minimization of functional

$$J(\{\chi\}) = \|[\psi]\{\chi\} - \{h\}\|^2 + \alpha_{reg}^2 \|[w]\{\chi\}\|^2 \tag{40}$$

For $\alpha_{reg} = 1$ minimization of functional (40) is identical with solution of system of equation (36). If exact solution $\{\chi^0\}$ is known then minimization of functional

$$J(\{\chi\}, \{\chi^0\}) = \|[\psi]\{\chi\} - \{h\}\|^2 + \alpha_{reg}^2 \left\|[w](\{\chi\} - \{\chi^0\})\right\|^2 \tag{41}$$

is identical with solution of system of equation for each values of parameter α_{reg} what leads to system of equations

$$\begin{bmatrix} [\chi] \\ \alpha_{reg}[w] \end{bmatrix} \{\chi\} = \begin{Bmatrix} \{h\} \\ \alpha_{reg}[w]\{\chi^0\} \end{Bmatrix} = \begin{Bmatrix} \{h\} \\ \{0\} \end{Bmatrix} + \begin{bmatrix} [0] \\ \alpha_{reg}[w] \end{bmatrix} \{\chi^0\} \tag{42}$$

or

$$\begin{gathered}[\psi_\alpha]\{\chi\}=\{F_1\}+[P_\alpha]\left\{\chi^0\right\}, \\ \dim[\psi_\alpha]=\dim[P_\alpha]=(M+1+M-2)\times(M+1)\end{gathered} \tag{43}$$

then

$$\begin{gathered}\{\chi\}=[\psi_\alpha]^+\cdot\{F\}+[\psi_\alpha]^+\cdot[P_\alpha]\left\{\chi^0\right\}=\{G_\alpha\}+[Q_\alpha]\left\{\chi^0\right\} \\ \dim[Q_\alpha]=(M+1)\times(M+1)\end{gathered} \tag{44}$$

In general case vector $\left\{\chi^0\right\}$ is unknown, then created iteration process

$$\left\{\chi^{n+1}\right\}=\{G_\alpha\}+[Q_\alpha]\{\chi^n\}, \quad n=0,1,2,... \tag{45}$$

we have

$$\left\{\chi^{n+1}\right\}=\sum_{j=0}^{n}[Q_\alpha]^j\cdot\{G_\alpha\}+[Q_\alpha]^{n+1}\left\{\chi^0\right\} \tag{46}$$

If spectral radius ρ_s of matrix $[Q_\alpha], \rho_s([Q_\alpha])<1$, then Neumann series in w (46) is convergent.

In case considered in the paper the spectral radius $\rho_s=1$.

For determination of regularization parameter α_{reg} a modification of L-curve [8] is used. As regularization matrix $[w]$ the matrix (36) resulting from condition (36) is taken. Classic L-curve is presented on Fig. 5, which correspond matrix $[w]=[I]$. For matrix $[w]$ determined according (39) this curve has shape given on Fig. 6. For acquisition explicit relationship with respect regularization parameter α_{reg} the function $\|w\cdot X\|/\|A\cdot X-h\|=f(\alpha)$ is introduced and presented on Fig. 7. The corner points on Figs 6 and 7 correspond the same value parameter α_{reg}. This value of parameter α_{reg} correspond the minimum non-dimensional function $\|R\cdot X\|/\max(\|R\cdot X-h\|)=f(\|A\cdot X-h\|/\max(\|A\cdot X-h\|))$ on Fig. 8, $[R]=\lfloor\alpha_{reg}\cdot w\rfloor$. On Fig. 8 non-dimensional values $\|X\|/\max(\|X\|)=f(\|A\cdot X-h\|/\max(\|A\cdot X-h\|))$, $\alpha_{reg}/\alpha_{reg-max}$, as a function of $f(\|A\cdot X-h\|/\max(\|A\cdot X-h\|))$ are given. The point of extreme on curve correspond the inflexion of curves. For optimal values regularization parameter α_{Reg} the inverse determination of temperature at points $\xi=0$ and $\xi=1$ and comparison with exact data was done. Obtained results confirm appropriate choice the curve $\|R\cdot X\|/\max(\|R\cdot X\|)=f(\|A\cdot X-h\|/\max(\|A\cdot X-h\|))$. This curve can be modified for obtained relationship $\|R\cdot X\|/\max(\|R\cdot X-h\|)/\max(\|R\cdot X\|/\|A\cdot X-h\|)=f(\alpha)$, what is presented on Fig. 9.

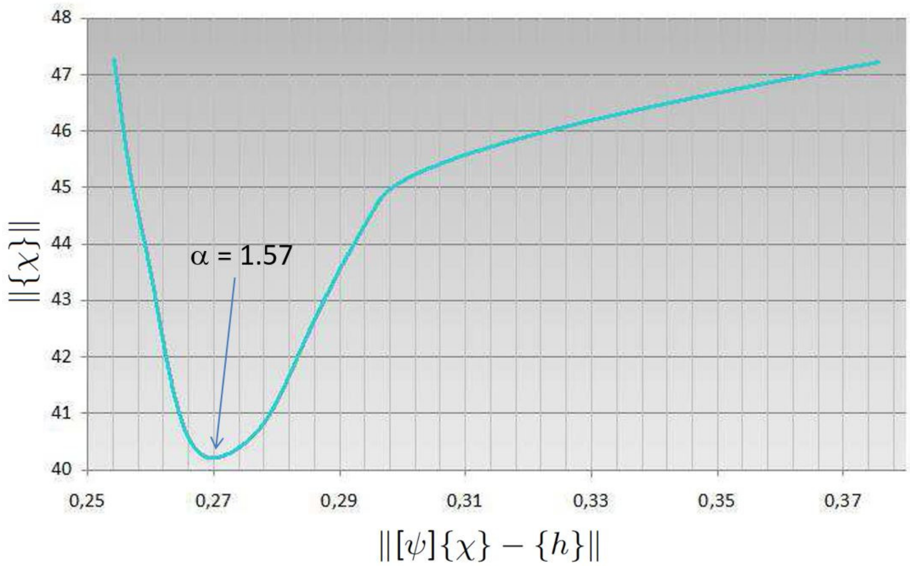


Fig. 5. Classic L - curve (Hansen[8]), $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$, $\min\|X\|$ is for $\alpha = 1.57$

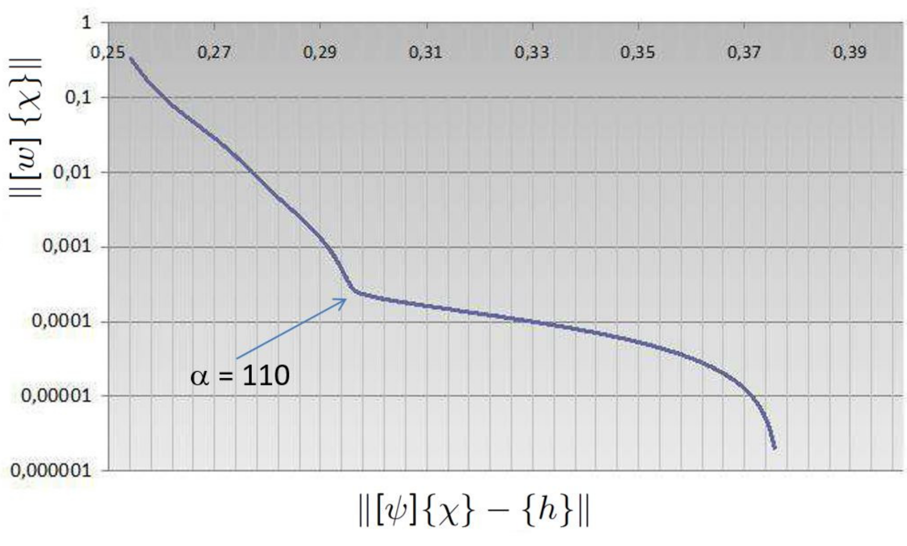


Fig. 6. L_w - curve, $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$, $\alpha_{corner} = 110$

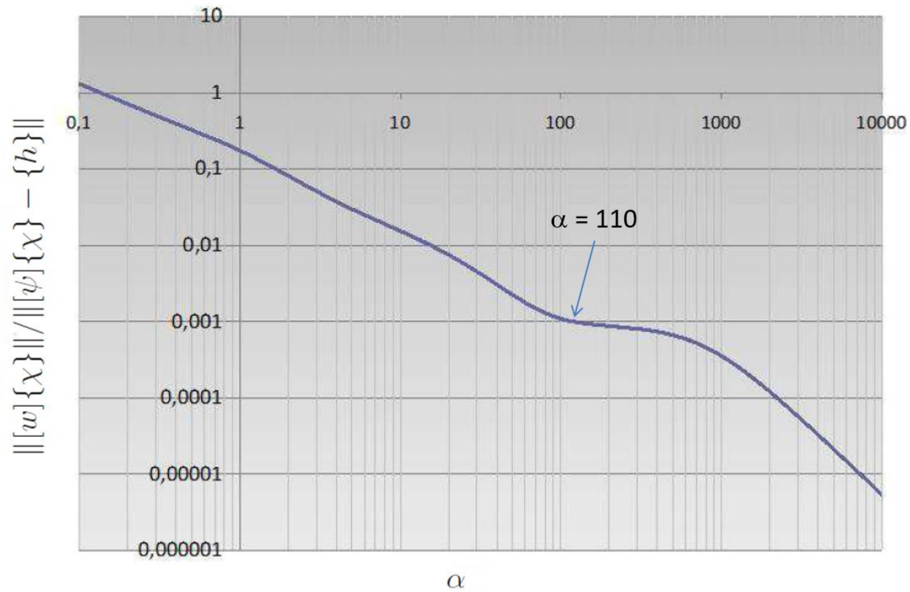


Fig. 7. L_w_α- curve, $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$

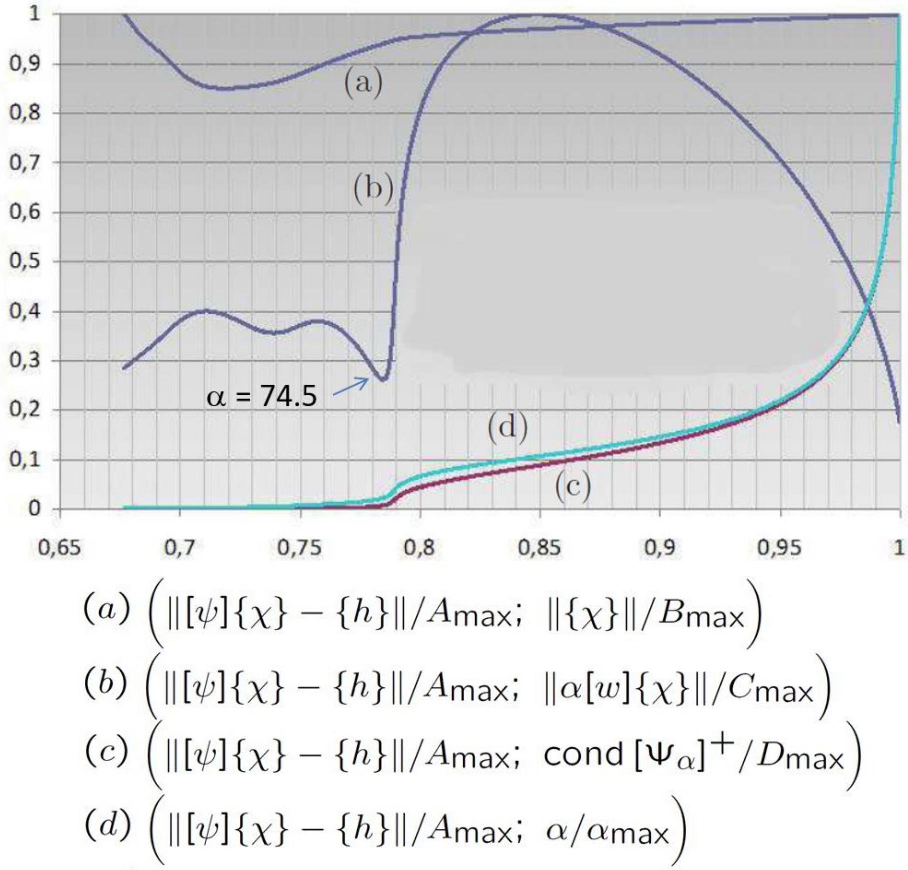
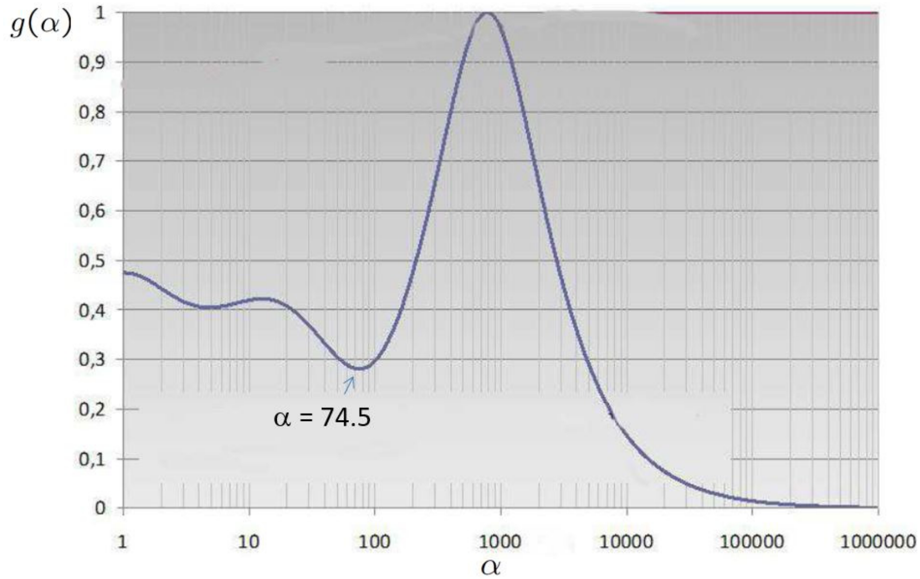


Fig. 8. Dimensionless distributions of: $\|R(\alpha)\cdot X\| \,/\, \max\left(\|R(\alpha)\|\right)$, α/α_{max}, $\|X\| \,/\, \max\|X\|$, $condn/condn_{max}$, $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$



$$g(\alpha) = \frac{\|\alpha[w]\{\chi\}\|}{\|[\psi]\{\chi\} - \{h\}\|} \left(\max_{\alpha \in [0, \alpha_{\max}]} \frac{\|\alpha[w]\{\chi\}\|}{\|[\psi]\{\chi\} - \{h\}\|} \right)^{-1}$$

Fig. 9. L_R_α- curve, $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$ $N_t = 200$, $\alpha_{reg\,opt} = 74.5$

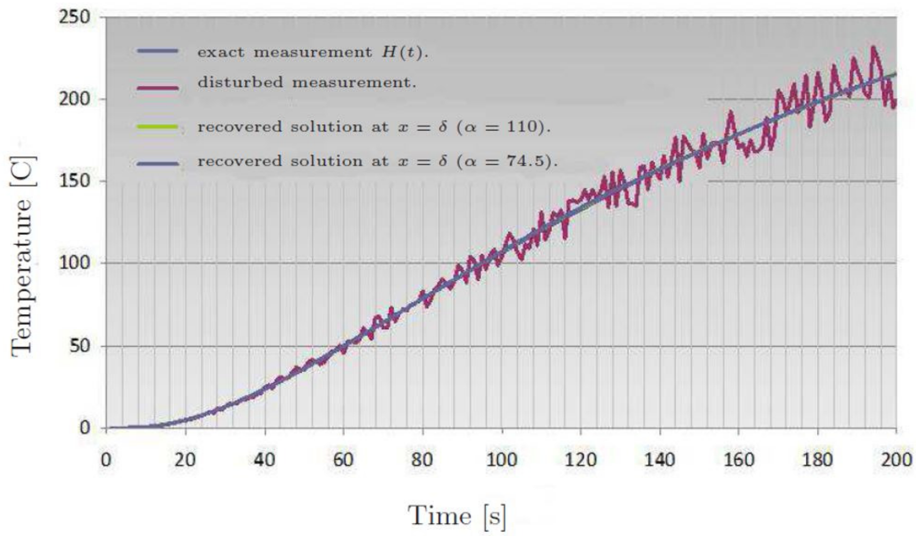


Fig. 10. Comparision of solution of inverse problem with given data, $I\_Matrix\_Reg = 0$, boundary $\xi = 1.0$, $\alpha_{reg} = 110$, $\alpha_{reg\ opt} = 74.5$

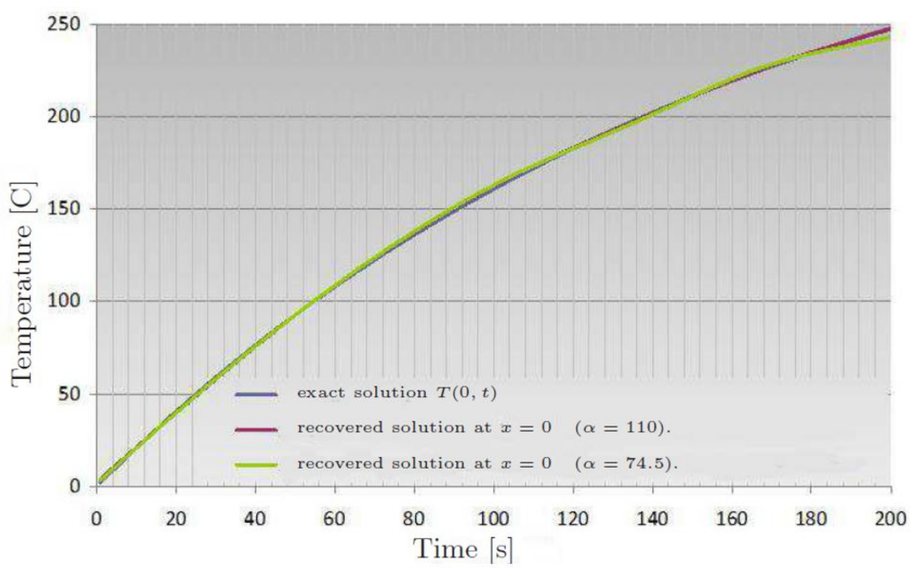


Fig. 11. Comparision of solution of inverse problem with exact data, $|\lambda_{random}| = 10\%$, $I\_Matrix\_Reg = 0$, boundary $\xi = 1.0$, $\alpha_{reg} = 110$, $\alpha_{reg\ opt} = 74.5$

4 Final Remarks

Consideration given in paper permit on replacement of classic L-curve (Hanson [8]) it's modification version L-w-curve, where matrix of regularization w is taking account (for $w = I$ the curve L-w is the same as L-curve). Optimal point for L-w-curve is very near of extremal point on L-R-curve (see Fig. 9). L-R-curve is function of parameter α and permit to keep track of change of $\|R(\alpha) \cdot X\| / \|A \cdot X - h\|$ as function of parameter α. Optimal point on L-R-curve corresponds point of inflexion of function $\|X\|$ and condition number condn of function $\|A \cdot X - h\|$, Fig. 8.

Tested in paper way of regularization given by (39) permit to obtained good results of Cauchy problem even if the error of measurement is big and equal $|\lambda_{random}| = 10.0\%$ it results from fact that chosen way of regularization to force smoothness of solution neat to physical distribution.

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