EPiC Series in Computer Science

Volume 36, 2015, Pages 27–41

GCAI 2015. Global Conference on Artificial Intelligence

A Modelling Pearl with Sortedness Constraints

Nicolas Beldiceanu[1], Mats Carlsson[2], Pierre Flener[3], Xavier Lorca[1], Justin Pearson[3], Thierry Petit[1,4], and Charles Prud'homme[1]

[1] TASC (CNRS/INRIA), Mines Nantes, FR – 44307 Nantes, France
FirstName.LastName@mines-nantes.fr
[2] SICS, P.O. Box 1263, SE – 164 29 Kista, Sweden
Mats.Carlsson@sics.se
[3] Uppsala University, Department of Information Technology, SE – 751 05 Uppsala, Sweden
FirstName.LastName@it.uu.se
[4] Foisie School of Business, Worcester Polytechnic Institute, Worcester, MA 01609, USA
TPetit@wpi.edu

Abstract

Some constraint programming solvers and constraint modelling languages feature the Sort(L, P, S) constraint, which holds if S is a nondecreasing rearrangement of the list L, the permutation being made explicit by the optional list P. However, such sortedness constraints do not seem to be used much in practice. We argue that reasons for this neglect are that it is impossible to require the underlying sort to be *stable*, so that Sort cannot be guaranteed to be a total-function constraint, and that L cannot contain *tuples* of variables, some of which form the key for the sort. To overcome these limitations, we introduce the StableKeysort constraint, decompose it using existing constraints, and propose a propagator. This new constraint enables a powerful modelling idiom, which we illustrate by elegant and scalable models of two problems that are otherwise hard to encode as constraint programs.

1 Motivation

In order to motivate our work, we consider the following human resource planning problem, which will serve as running example throughout this paper. Consider n tasks that are to be assigned to a given set of m employees. For each task $i \in 1..n$, let attribute A_i denote the initially unknown employee performing it, B_i its fixed beginning time, D_i its fixed duration, and E_i its fixed end time, with $E_i = B_i + D_i$ for all $i \in 1..n$. A first constraint is that tasks assigned to the same employee should not overlap in time. A second constraint deals with the allocation of breaks to each employee. This is done by stating a Shift constraint, which we now describe. A *shift* is a maximum sequence of tasks assigned to a same employee such that the time gap between any two consecutive tasks is shorter than a given threshold *minBreak*: that is, consecutive shifts of an employee are separated by a *break* of minimum duration *minBreak*, and consecutive tasks of a shift are separated by a gap shorter than *minBreak*. The *span* of a shift is the difference between the end time of its last task and the beginning time of its first task: see Figure 1. The Shift constraint holds if and only if the span of each shift of each employee is at most a given threshold *maxSpan*.

G.Gottlob, G.Sutcliffe and A.Voronkov (eds.), GCAI 2015 (EPiC Series in Computer Science, vol. 36), pp. 27–41

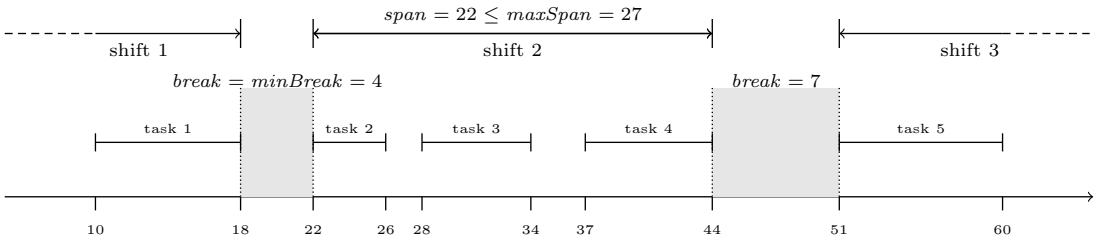


Figure 1: Shift 2 of an employee is composed of tasks 2, 3, and 4. It is between shifts 1 and 3 of the same employee. A break of duration 4 is scheduled between task 1 of shift 1 and task 2 of shift 2, because the gap between these tasks is at least $minBreak = 4$. Similarly, a break of duration 7 is scheduled after task 4 of shift 2 and task 5 of shift 3. No other breaks can be scheduled between the tasks of shift 2 because of the minimum break duration $minBreak$. The span of shift 2 is 22 and does not exceed $maxSpan = 27$: it is composed of tasks 2, 3, and 4, as well as of the two gaps between these tasks.

Concisely expressing constraints like Shift is hard. We show that sortedness constraints, possibly upon some extensions, enable elegant models of such constraints and problems.

The *contributions* and *organisation* of the rest of this paper are as follows. In Section 2, we discuss the existing sortedness constraints and their limitations. In Section 3, we introduce the StableKeysort constraint, which overcomes those limitations. In Section 4, we argue that composing the new StableKeysort constraint with other constraints is a powerful modelling idiom. We illustrate this idiom by elegant models of the motivating problem of this introduction and another problem. We empirically evaluate our results in Section 5 and conclude in Section 6.

2 The Sort Constraints and Their Limitations

Constraint programming (CP, e.g., [23]) is a problem solving technology where relations between *variables* (or: unknowns) are stated in the form of *constraints*, which together form a *model* of a constraint problem. Each variable x has a finite *domain* of possible values, usually a set of integers, whose lower and upper bounds are here respectively denoted by $\underline{x}$ and $\overline{x}$. Each constraint has an associated *propagator*, which removes domain values that cannot be part of a solution to that constraint. Depending on the propagators used, the domain reduction for all constraints of the model, called *propagation*, can be more or less effective: the notion of *consistency* characterises propagator effectiveness. Some propagators only adjust the domain bounds, achieving *bounds*($\mathbb{Z}$) *consistency* [12] for example. A propagator that only keeps domain values that participate in a solution to its constraint is said to achieve *domain consistency* [23]. A *solution to a constraint* is obtained when the domains of all its variables are singletons: we say that the variables are *fixed*. If propagation alone does not lead to a *solution to the constraint problem*, that is to a solution to each of its constraints, then *search* is interleaved with further propagation at every node of the search tree, with *backtracking* whenever some domain is shrunk to the empty set. A constraint, especially when no propagator is available for it in the used CP solver, can be represented by a conjunction of other constraints, called a *decomposition* of the constraint.

The Sort(L, S) constraint was introduced, without exhibiting a propagator, by Older *et*

al. [17] for expressing disjunctive constraints in job-shop scheduling problems: it holds if and only if the list S of variables is a nondecreasing rearrangement of the list L of variables. Also motivated by the job-shop problem, Zhou [30, 31] introduced, also without exhibiting a propagator, the $\textsc{Sort}(L, P^{-1}, S)$ constraint, where the permutation P^{-1} of the integer interval $1..n$ linking the variables of S to those of L is made explicit: we have $L_i = S_{P_i^{-1}}$ for all positions i. (Our reason for referring to the permutation as P^{-1} will become clear in Section 3.1.) For example, $\textsc{Sort}([5,4,4],[3,1,2],[4,4,5])$ holds. Introducing the variables of P^{-1} and fixing them during *search* is convenient for expressing a branching heuristic that selects which task is scheduled before or after a group of tasks, without explicitly adding precedence constraints during search [30, 31]. A *modelling* use of P^{-1} will be given at the end of Section 3.2.

Bleuzen-Guernalec and Colmerauer [5, 6] proposed the first bounds($\mathbb{Z}$) consistency [12] propagator for $\textsc{Sort}(L, S)$: it has a worst-case time complexity of $O(n \log n)$, where n is the length of the lists L and S. When sorting can be achieved in linear time, an $O(n)$ bounds($\mathbb{Z}$) consistency propagator for $\textsc{Sort}(L, S)$ was introduced by Mehlhorn and Thiel [15], maintaining convexity [14] of the underlying bipartite graph. It was generalised [27] for the $\textsc{Sort}(L, P^{-1}, S)$ constraint of SICStus [11, 10], ECLiPSe [26], and Gecode [13, 19], at the expense of bounds consistency. The latter two provide both versions, Choco [21] and or-tools [18] only provide the two-argument version, whereas all other CP solvers we checked provide neither of them.

Even if the Sort constraint, with or without the permutation variables, is available in a few CP solvers, as well as in the constraint modelling language MiniZinc [16], it does not seem to be used much. Beyond the original papers, we found very few published uses of Sort, notably [2, 8], and very few uses in public repositories of constraint models. In [7], a mathematical model using Sort is given for a meeting scheduling problem, but the translation into MiniZinc overlooks the Sort constraint of MiniZinc. The $\textsc{Sort}(L, S)$ constraint was mentioned in [4] as a way to help decompose constraints. For example, the constraint $\textsc{NValue}([v_1, \ldots, v_n], N)$, which holds if and only if there are N distinct values in the sequence $[v_1, \ldots, v_n]$, can be decomposed by constraining N to be the number of *strict* increases between adjacent variables in the sorted rearrangement of $[v_1, \ldots, v_n]$. While this is an important observation, the use of the Sort constraint for decomposing constraints reveals two limitations that we discuss now.

2.1 The $\textsc{Sort}(L, P^{-1}, S)$ Constraint Is Not a Total-Function Constraint

The $\textsc{Sort}(L, P^{-1}, S)$ constraint is not a total-function constraint [4], as a ground list L does not uniquely determine P^{-1} and S. For example, $\textsc{Sort}([5,4,4],[3,1,2],[4,4,5])$ holds, but $\textsc{Sort}([5,4,4],[3,2,1],[4,4,5])$ also holds, with a different permutation for the duplicate elements inside L and S. The problem is that the $\textsc{Sort}(L, P^{-1}, S)$ constraint does not require the sort to be *stable*, that is to use the position of a variable in L in order to break ties deterministically. Not uniquely determining P^{-1} from a list L with duplicates is problematic for constraint decompositions involving $\textsc{Sort}(L, P^{-1}, S)$ where the variables of P^{-1} are used in other constraints whose propagation is hindered until all variables of P^{-1} are fixed. For example, $\textsc{Sort}([5,4,4],[P_1^{-1}, P_2^{-1}, P_3^{-1}],[S_1, S_2, S_3])$ yields $S_1 = 4$, $S_2 = 4$, $S_3 = 5$, $P_1^{-1} = 3$, $P_2^{-1} \in 1..2$, and $P_3^{-1} \in 1..2$: the last two variables are not fixed. In Section 4.2, we present a model where stability is crucial for effective propagation.

To overcome this difficulty, the modeller must be aware of the variables of P^{-1} when they are not part of the overall problem. While this is possible, the effect is that the decomposition cannot be used in a transparent way, as the variables of P^{-1} must be considered during search. Determining when to fix the variables of P^{-1} during search is not straightforward: on the one hand, fixing them too late may be inefficient since subproblems may be recognised belatedly as

infeasible when these variables are not fixed; on the other hand, fixing them too early and in a wrong order may lead to a lot of backtracks.

In this paper, we argue that it is preferable to make $\textsc{Sort}(L, P^{-1}, S)$ a total-function constraint, based on a stable sort.

2.2 The Sort Constraints Cannot Handle Lists of Keyed Tuples

Quite often, rather than constraining a sorted list of variables, one needs to constrain a lexicographically sorted list of *tuples* of variables, where each tuple typically corresponds to the attributes of an object, some of which form the sorting key. Without loss of generality, we assume the first $k \geq 1$ fields of a tuple form the sorting key.

For example, a task i of the motivating problem in Section 1 yields a tuple $\langle A_i, B_i, D_i, E_i \rangle$ of $q = 4$ fields: modelling the Shift constraint is facilitated when constraining the list of task tuples that is lexicographically sorted on the key $\langle A_i, B_i \rangle$ of their first $k = 2$ fields, thereby bundling the tasks assigned to a same employee and ordering them by increasing beginning time, so that the shifts of each employee become apparent. The full details of this decomposition will be given in Section 4.1.

In this paper, we argue for the generalisation of Sort to a constraint on lists of keyed tuples.

3 The StableKeysort constraint

To overcome the limitations of the existing Sort constraints, as outlined in Section 2, we now introduce the StableKeysort constraint (Section 3.1), decompose it using existing constraints (Section 3.2), and discuss a dedicated propagator (Section 3.3).

3.1 Specification of StableKeysort

The $\textsc{StableKeysort}(L, P, S, k)$ constraint, where L and S are lists of $n \geq 0$ tuples of $q \geq 1$ variables per tuple, P is an optional list of n variables, and k is an integer value in $1..q$, holds if and only if the following three conditions hold:

1. All variables of P are assigned distinct values in the interval $1..n$.
2. The tuples of S correspond to those of L according to the permutation P: we have $S_i = L_{P_i}$ for all positions i. Note that we here prefer, for ease of modelling the decompositions below, to define P as the *inverse* of P^{-1} in $\textsc{Sort}(L, P^{-1}, S)$, where $L_i = S_{P_i^{-1}}$ for all i (see Section 2).
3. The tuples of S form a rearrangement of L that is stably sorted by nondecreasing lexicographical order on the first k positions in the tuples.

If the permutation argument P is omitted, then only condition 3 needs to hold. As we will see in Section 4, (the inverse of) P is often unnecessary for *modelling* purposes when one uses StableKeysort instead of Sort, so that the main use of P may lie in expressing *search* heuristics.

3.2 Decomposition of StableKeysort

We now give a decomposition of $\textsc{StableKeysort}(L, P, S, k)$ in terms of Sort, Element, and arithmetic constraints. The idea is based on having a unique key K_i that is a linear combination of the k first fields of tuple L_i as well as position i itself, for stability: the permutation P linking

$$\textsc{StableKeysort}([L_1,\ldots,L_n],[P_1,\ldots,P_n],[S_1,\ldots,S_n],k) \Leftrightarrow (1)\wedge(2)\wedge(3)\wedge(4)$$

$$K_i = \sum_{j\in 1..k} c_j \cdot L_{i,j} + i - 1,\ \forall i \in 1..n \tag{1}$$

$$\textsc{Sort}([K_1,\ldots,K_n],[K'_1,\ldots,K'_n]) \tag{2}$$

$$P_i = (K'_i \bmod n) + 1,\ \forall i \in 1..n \tag{3}$$

$$S_{i,j} = L_{P_i,j},\ \forall i \in 1..n,\ \forall j \in 1..q \tag{4}$$

where the constant bounds ℓ_j and u_j and the constant coefficients c_j are determined as follows:

$$\ell_j = \min\{\underline{L_{i,j}} \mid i \in 1..n\},\ \forall j \in 1..k$$

$$u_j = \max\{\overline{L_{i,j}} \mid i \in 1..n\},\ \forall j \in 1..k$$

$$c_j = \begin{cases} (u_{j+1} - \ell_{j+1} + 1) \cdot c_{j+1} & \text{if } 1 \le j < k \\ n & \text{if } j = k \end{cases}$$

Decomposition 1: Decomposition of the StableKeysort constraint.

the list K of keys K_i to its sorted rearrangement K' is also the permutation linking L to S. The decomposition computes the constants ℓ_j and u_j for the smallest respectively largest domain values in key position j, and introduces the auxiliary variables c_j for the coefficients of the linear combination, as well as the auxiliary variables K_i and K'_i as mentioned above. Decomposition 1, where the notation $X_{i,j}$ is used for field j of tuple i of list X, has drawbacks that motivate the need for a dedicated propagator for StableKeysort (see Section 3.3):

- The coefficients c_j can become quite large if the key fields have large domains, or if there are many of them. This can lead to overflow problems.
- For (1) and (3), standard CP solvers will not even achieve bounds($\mathbb{Z}$) consistency. We could achieve domain consistency for (1) and (3) with dynamic programming [28], but that can be very costly, again since the coefficients can be quite large.

Using the remark on condition 2 of the specification of StableKeysort, the constraints (2) and (3) can be reformulated without arithmetic as

$$\begin{array}{c}\textsc{Sort}([K_1,\ldots,K_n],[P_1^{-1},\ldots,P_n^{-1}],[K'_1,\ldots,K'_n]) \\ \wedge\ \textsc{Inverse}([P_1,\ldots,P_n],[P_1^{-1},\ldots,P_n^{-1}])\end{array}$$

when the constraint $\textsc{Sort}(L,P^{-1},S)$ is available, and where the constraint $\textsc{Inverse}(P,P^{-1})$ holds if $P_i = j \Leftrightarrow P_j^{-1} = i$, for all $i,j \in 1..n$.

3.3 Propagator for StableKeysort

We designed and implemented a propagator for StableKeysort, using the Mehlhorn and Thiel propagator [15] (MT) as a starting point. Our propagator runs in $O(nk \log nk) + O(nq)$

time. It does not guarantee bounds($\mathbb{Z}$) consistency, as shown at the end of this section. Let key(L_i) denote the *key part* of tuple L_i, consisting of the first k fields of L_i extended by i. Let key(S_i) denote the *key part* of tuple S_i, consisting of the first k fields of S_i extended by P_i. The function of each last position is twofold: it breaks ties among otherwise equal keys, ensuring stability, and it allows P to be obtained from the sorted key parts.

The core of MT for Sort(L, S) is the construction of a bipartite intersection graph, with an arc from L_i to S_j if $\underline{L_i}..\overline{L_i}$ and $\underline{S_j}..\overline{S_j}$ have a nonempty intersection. Then, the propagator computes a perfect matching, a reduced intersection graph, and its strongly connected components (SCCs), which are used for pruning the domains of the variables of L. Throughout the algorithm, the domains of the variables of S are kept *normalised*, i.e., $\underline{S_i} \leq \underline{S_{i+1}}$ and $\overline{S_i} \leq \overline{S_{i+1}}$ for all i. Normalisation allows MT to use a convex intersection graph, which admits a compact representation and linear-time operations. For details, see Section 1.2 of [15].

Our propagator for StableKeysort is an adaption of MT, using exactly the same phases, followed by one additional phase. In the following, if x is a tuple of variables, then $\underline{x}$ and $\overline{x}$ respectively denote the tuple of lower bounds and the tuple of upper bounds of the domains of those variables. Whenever MT uses a scalar comparison among $\underline{L_i}$, $\overline{L_i}$, $\underline{S_j}$, and $\overline{S_j}$, for example when sorting L on lower and upper bounds, during normalisation, or during SCC computation, we use a lexicographic comparison among $\underline{\text{key}(L_i)}$, $\overline{\text{key}(L_i)}$, $\underline{\text{key}(S_j)}$, and $\overline{\text{key}(S_j)}$. Whenever MT adjusts a lower bound, enforcing $L_i \geq c$, we enforce $\text{key}(L_i) \geq_{\text{lex}} c$, using methods from [9], and similarly for S_j and upper bounds. Our intersection graph contains an arc from L_i to S_j if:

$$\underline{\text{key}(L_i)} \leq_{\text{lex}} \overline{\text{key}(S_j)} \wedge \underline{\text{key}(S_j)} \leq_{\text{lex}} \overline{\text{key}(L_i)}.$$

Thus, as described so far, our propagator generalises MT by replacing scalar operations on the bounds of L_i and S_j by lexicographic operations on the bounds of key(L_i) and key(S_j). This, however, ignores the domains of the individual tuple fields and of the variables of P. This is why we need an additional pruning phase, to ensure that every tuple of L can be equal to at least one tuple of S in the same SCC, and vice versa. The additional phase removes infeasible values in four steps, for all $i \in 1..n$ and $j \in 1..q$, where $i \sim j$ denotes that L_i and S_j belong to the same SCC:

1. The bounds of any position j of any tuple of S must be inside the bounds of the same position j of the tuples of L. That is, enforce:

$$\overline{S_{i,j}} \leq \max\{\overline{L_{p,j}} \mid p \in 1..n \wedge p \sim i\}$$
$$\underline{S_{i,j}} \geq \min\{\underline{L_{p,j}} \mid p \in 1..n \wedge p \sim i\}$$

2. The bounds of any position j of any tuple of L must be inside the bounds of the same position j of the tuples of S. That is, enforce:

$$\overline{L_{i,j}} \leq \max\{\overline{S_{p,j}} \mid p \in 1..n \wedge i \sim p\}$$
$$\underline{L_{i,j}} \geq \min\{\underline{S_{p,j}} \mid p \in 1..n \wedge i \sim p\}$$

3. The bounds of any permutation variable must be inside the possible tuple positions. That is, enforce:

$$\overline{P_i} \leq \max\{p \mid p \in 1..n \wedge p \sim i\}$$
$$\underline{P_i} \geq \min\{p \mid p \in 1..n \wedge p \sim i\}$$

4. Any tuple position must be inside the bounds of the permutation variables. That is, check:

$$i \leq \max\{\overline{P_p} \mid p \in 1..n \wedge i \sim p\}$$
$$i \geq \min\{\underline{P_p} \mid p \in 1..n \wedge i \sim p\}$$

The phases that were generalised from MT take $O(nk)$ time, except for sorting L, which takes $O(nk \log nk)$ time. The additional phase takes $O(nq)$ time, yielding a total time complexity of $O(nk \log nk) + O(nq)$.

The fact that the matching ignores the domains of individual tuple positions and of permutation variables is the main weakness of our propagator and the main reason why it sometimes does not achieve bounds($\mathbb{Z}$) consistency. For example:

$$x, y, z \in 1..3 \wedge \textsc{StableKeysort}([[x, 2], [y, 3]], [[2, 3], [z, 2]], 1)$$

will prune the domains to $x, y, z \in 2..3$ instead of $x = 3, y = 2, z = 3$, which is the only solution. Unfortunately, taking domain information into account would mean losing the convexity of the bipartite graph, but convexity is the property that allows all phases of MT except the initial sort to run in linear time.

Achieving domain consistency for StableKeysort is NP-hard [24], but is more interesting than achieving bounds consistency, as fields of the key can be attributes such as an employee identifier, for which bounds propagation is meaningless. The tractability of achieving bounds($\mathbb{Z}$) consistency for $\textsc{Sort}(L, P^{-1}, S)$ and StableKeysort is still open [24].

4 A Modelling Pearl with StableKeysort

Our motivation for introducing the StableKeysort constraint, with or without the permutation argument, is to maintain the relation between a given list of tuples and the corresponding lexicographically sorted tuples on the k first positions of the given tuples, so that a constraint on the given tuples can be expressed on the sorted tuples instead. Typically, by scanning the sorted tuples we can directly express the required constraint. In other words, the modelling idiom that we want to convey is:

$$\mathit{Constraint}(L) \Leftrightarrow \textsc{StableKeysort}(L, S, k) \wedge \mathit{ConstraintOnSequence}(S)$$

where it is awkward or unknown how to express *Constraint* by other means, but well known how to express *ConstraintOnSequence*. We now show two instances of this modelling pearl.

4.1 The Shift Constraint

We now revisit the motivating problem of Section 1 and model Shift as a decomposition into StableKeysort and simple arithmetical and logical constraints: see Decomposition 2. The decomposition introduces the auxiliary variables A'_i, B'_i, D'_i, E'_i for a sorted list of task attribute tuples ($q = 4$), as well as auxiliary 0..1 variables Y_i, denoting whether two consecutive tasks $i - 1$ and i are performed by the same employee, and X_i, denoting whether task i is a subsequent task of the current shift. A last set of auxiliary variables are the R_i over domain $1..\mathit{maxSpan}$, denoting the shift length up to the end of task i. In the sortedness constraint (6), the permutation argument P is not needed, and stability is not needed (since the tasks assigned to a same employee are not allowed to overlap, so that their beginning times break ties), but crucial use is made of tuples, keyed on the assigned employee identity A_i and the beginning time

$$\textsc{Shift}([\langle A_i, B_i, D_i, E_i\rangle \mid i \in 1..n], [\langle A'_i, B'_i, D'_i, E'_i, Y_i\rangle \mid i \in 1..n], \\ \mathit{minBreak}, \mathit{maxSpan}) \Leftrightarrow (5) \wedge (6) \wedge (7) \wedge (8) \wedge (9) \wedge (10)$$

$$E_i = B_i + D_i, \ \forall i \in 1..n \tag{5}$$

$$\textsc{StableKeysort}([\langle A_i, B_i, D_i, E_i\rangle \mid i \in 1..n], [\langle A'_i, B'_i, D'_i, E'_i\rangle \mid i \in 1..n], 2) \tag{6}$$

$$A'_{i-1} < A'_i \ \vee \ E'_{i-1} \leq B'_i, \ \forall i \in 2..n \tag{7}$$

$$Y_i = 1 \Leftrightarrow \begin{cases} \textbf{false} & \text{if } i = 1 \\ A'_i = A'_{i-1} & \text{if } i \in 2..n \end{cases} \tag{8}$$

$$X_i = 1 \Leftrightarrow \begin{cases} \textbf{false} & \text{if } i = 1 \\ Y_i \ \wedge \ B'_i - E'_{i-1} < \mathit{minBreak} & \text{if } i \in 2..n \end{cases} \tag{9}$$

$$R_i = \begin{cases} D'_i & \text{if } i = 1 \\ D'_i + X_i \cdot (R_{i-1} + B'_i - E'_{i-1}) & \text{if } i \in 2..n \end{cases} \tag{10}$$

where

$$R_i \in 1..\mathit{maxSpan}, \ \forall i \in 1..n$$

Decomposition 2: Decomposition of the SHIFT constraint.

B_i of each task ($k = 2$). Constraint (7) prevents two consecutive tasks with the same employee from overlapping. Constraints (9) and (10) channel between the sorted task attributes and the X_i and R_i.

4.2 The CUMULATIVESMOOTH Constraint

The cumulative scheduling problem (CuSP) is frequently modelled using the CUMULATIVE(T, c) constraint [1]. A CuSP is defined on a set T of n tasks consuming the same resource, with a fixed integer capacity, c, of the resource. Each task $i \in 1..n$ is here defined by the following attributes: its beginning time B_i, its duration D_i, its end time E_i, with $E_i = B_i + D_i$, and its height H_i, that is the quantity of resource necessary to perform it. Tasks may overlap in time. A solution to a CuSP is a schedule that satisfies the following constraint, requiring that the resource capacity never be exceeded:

$$\sum_{\substack{i \in 1..n: \\ B_i \leq t < E_i}} H_i \leq c, \ \forall t \in \mathbb{Z}$$

In many CuSP applications, in addition to the resource capacity, the profile of resource consumption throughout the schedule is constrained: a frequent constraint is a limit on the number of large resource consumption variations between consecutive time points in this profile. For example, teams often cannot vary at arbitrary times in personnel scheduling: the human resource profile needs to be smoothened; other examples are in [29, Section 5.2]. The SMOOTH($N, [X_1, X_2, \ldots, X_n], \tau$) constraint [3] holds if and only if $|X_i - X_{i+1}| > \tau$ is satisfied

$$\textsc{CumulativeSmooth}([\langle B_i, D_i, E_i, H_i\rangle \mid i \in 1..n], c, N, \tau) \Leftrightarrow (11) \wedge (12) \wedge (13) \wedge (14) \wedge (15)$$

$$\textsc{Cumulative}([\langle B_i, D_i, E_i, H_i\rangle \mid i \in 1..n], c) \tag{11}$$

$$\textsc{StableKeysort}([\langle B_i, H_i\rangle, \langle E_i, -H_i\rangle \mid i \in 1..n], [\langle \mathit{Instant}_i, \Delta_i\rangle \mid i \in 1..2n], 1) \tag{12}$$

$$\mathit{Current}_i = \begin{cases} 0 & \text{if } i = 0 \vee i = 2n \\ \mathit{Current}_{i-1} + \Delta_i & \text{if } 1 \leq i < 2n \end{cases} \tag{13}$$

$$\mathit{Profile}_i = \begin{cases} 0 & \text{if } i = 0 \vee i = 2n \\ \mathit{Profile}_{i+1} & \text{if } 1 \leq i < 2n \wedge \mathit{Instant}_i = \mathit{Instant}_{i+1} \\ \mathit{Current}_i & \text{if } 1 \leq i < 2n \wedge \mathit{Instant}_i < \mathit{Instant}_{i+1} \end{cases} \tag{14}$$

$$\textsc{Smooth}(N, [\mathit{Profile}_i \mid i \in 0..2n], \tau) \tag{15}$$

Decomposition 3: Decomposition of the CumulativeSmooth constraint.

N times for $1 \leq i < n$, where N is an integer variable and τ a positive integer. Unfortunately, Cumulative usually does not provide variables representing the resource profile, although the ECLiPSe [26] constraint Profile does, so in most CP solvers, we cannot use Smooth in a CuSP context. In order to address this issue, we introduce the CumulativeSmooth(T, c, N, τ) constraint, which combines a Cumulative constraint with a Smooth constraint on variables representing the profile. At time t, the *profile height* is the accumulated height, that is $\sum_{i \in 1..n : B_i \leq t < E_i} H_i$. Such a profile is built in three steps:

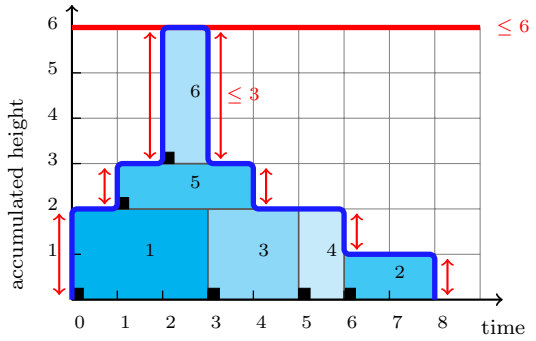
1. Construct for each task two pairs ($q = 2$): one with the task beginning time and height, the other with the task end time and negated height: see the first argument of the StableKeysort constraint (12).

2. Sort these pairs on their first item ($k = 1$), i.e., on the beginning and end times of the tasks: see constraint (12).

3. Build the profile from the sorted list obtained in phase 2: see constraints (13) and (14).

The profile height variables admit the statement of a Smooth constraint on the number of big variations between consecutive time points: see constraint (15). All this leads to a decomposition of CumulativeSmooth that uses StableKeysort: see Decomposition 3 and note how all arithmetic constraints are elegantly stated by reasoning on the *sorted* lists of variable tuples. In the sortedness constraint (12), the permutation argument P is not needed, but crucial use is made of keyed tuples and stability: without stability, the Δ_i associated to identical instants in constraint (12) will only be fixed during search, leading to poor propagation.

Figure 2 shows an example CuSP, the variables that arise from the decomposition of its CumulativeSmooth instance, and their values.



i	0	1	2	3	4	5	6	7	8	9	10	11	12
B_i		0	6	3		5	1	2					
D_i		3	2	2		1	3	1					
E_i		3	8	5		6	4	3					
H_i		2	1	2		2	1	3					
$Instant_i$	0	1	2		3	3		3	4	5	5	6	6	8
Δ_i		2	1	3	−2	2	−3	−1	−2	2	1	−2	−1	
$Current_i$	0	2	3	6		4	6	3	2	0	2	3	1	0
$Profile_i$	0	2	3	6		3	3	3	2	2	2	1	1	0

Figure 2: A cumulative scheduling problem and the values of the variables that arise from the decomposition of CumulativeSmooth($[\langle B_i, D_i, E_i, H_i\rangle \mid i \in 1..6], 6, 0, 3$): the cumulative profile should never exceed $c = 6$ and the profile variation (shown by vertical arrows) should never ($N = 0$) exceed $\tau = 3$.

4.3 Synthesis

Both the Shift and CumulativeSmooth examples can be seen as typical instances of problems in which one needs to generate profiles represented by *sorted* events where side constraints, such as work regulation rules for employees in the case of Shift, or preferences on the resource profile in the case of CumulativeSmooth, should be added to the model before search. As work regulation rules and preferences on the resource profile can vary a lot, depending on the peculiarity of the application, a modelling idiom makes sense since implementing a dedicated propagator for each variant would be very time consuming. One may observe that Decomposition 3, for instance, can easily be generalised: the Smooth constraint (15) can be replaced by other constraints, such as a balancing constraint [25]. Note also that our modelling idiom leads to constraint decompositions that are *linear* in memory with respect to the number of tasks.

5 Empirical Evaluation

For an empirical evaluation, we revisit the human resource planning problem introduced in Section 1. The time line is split into t 15-minute time slots. The objective is to minimise labour cost, subject to the Shift constraint as defined in Section 4.1.Let *dayDuration* denote the maximum amount of time slots per day for an employee, and let $cost_{a,w}$ denote the cost of employee a when at work for w time slots in one day.

An elegant formulation, with the Shift constraint, of the problem is provided as Model 1. The working time of the employees are maintained (16) in order to compute their costs (17). In our evaluation, Shift is implemented in terms of the proposed new StableKeysort constraint as shown in Decomposition 2.

A less elegant formulation, without the Shift constraint, of the problem is provided as Model 2. Let variable $a_{j,k}$ denote the activity of employee j in slot k: a task is represented by its index from 1 to n; a break shorter than *minBreak* is represented by the special value $n+1$; a long break (between *minBreak* and *maxSpan*) is represented by the special value $n+2$; and any other inactivity is represented by the special value $n+3$. Constraint (18) applies to the sequence of $a_{j,k}$ for each employee j, where the Regular(X, A) constraint [20] holds if and only if the

Minimise

$$\sum_{j=1}^{n} C_j$$

subject to

$$\textsc{Shift}([\langle A_i, B_i, D_i, E_i\rangle \mid i \in 1..n], [\langle A'_i, B'_i, D'_i, E'_i, Y_i\rangle \mid i \in 1..n], minBreak, maxSpan)$$

$$W_j = \begin{cases} D'_j & \text{if } j = 1 \\ D'_j + Y_j \cdot (W_{j-1} + B'_j - E'_{j-1}) & \text{if } j \in 2..n \end{cases} \tag{16}$$

$$C_j = \begin{cases} (1 - Y_{j+1}) \cdot cost_{A'_j, W_j} & \text{if } j \in 1..n-1 \\ cost_{A'_j, W_j} & \text{if } j = n \end{cases} \tag{17}$$

where

$$W_j \in 0..dayDuration, \ \forall j \in 1..n$$

Model 1: Formulation of the shift problem with the SHIFT constraint.

word represented by the sequence X of variables is accepted by the given finite automaton A, where double circles denote accepting states. Here, the automaton is the intersection of one automaton encoding the break rules (depicted in the inner box of Model 2) and one automaton Π encoding the task successor relation, which stems from the fixed beginning and end times. Constraints (19) and (20) ensure that, for each slot, every task is performed by at most one employee, where the GCC(X, V, O) constraint [22] holds if and only if there are O_i occurrences of value V_i within the sequence X of variables. The non-inactivities of the employees serve to compute their working times, via constraints (21) and (22), and their costs, with 0..1 variable $w_{j,k}$ denoting whether employee j is incurring cost in time slot k.

We compare Model 2 against two implementations of Model 1, namely one where the required STABLEKEYSORT constraint is implemented by Decomposition 1, and one where it is implemented by the propagator of Section 3.3. As stated in Section 3.2, we expect the implementation based on Decomposition 1 to produce integer overflows for problem instances where the key fields (namely the employee and the beginning time of a task) have large domains. Neither implementation of STABLEKEYSORT will even achieve bounds($\mathbb{Z}$) consistency, as noted in Section 3.2 and Section 3.3, respectively, whereas domain consistency can be achieved on REGULAR and GCC.

The used real-world problem instances involve up to 3,200 tasks and originate from Eurodecision and express a bus driver planning problem. The models above are simplified in order to hide industrial constraints that fall within the competitive advantage of Eurodecision. The experiments were run under Choco [21] on a Mac Pro with 8-core Intel Xeon E5 at 3 GHz under MacOS 10.10 and Java 1.8.0_25. Each instance was run with a 30-minute limit on its own core, each with up to 4 GB of memory. For every model, the search heuristic tries to assign the employees one by one, in lexicographic index order, to the most suitable free task, considering their current workload. This implies that the first solutions are the same for all models, but alternative solutions may come in a different order.

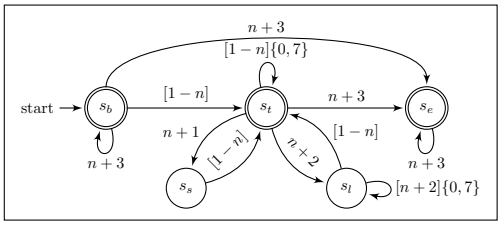
The results are given in Table 1, where "out of memory" denotes that the Java Virtual Machine ran out of memory and could not allocate more memory, while "integer overflow"

Minimise

$$\sum_{j=1}^{m} cost_{j,W_j}$$

subject to

$$\textsc{Regular}([a_{j,k} \mid k \in 1..t], \; \& \, \Pi), \; \forall j \in 1..m \tag{18}$$



$$\text{GCC}([a_{j,k} \mid j \in 1..m], [1, \ldots, n+3], [O_1, \ldots, O_{n+3}]), \; \forall k \in 1..t \tag{19}$$

$$O_i \leq 1, \; \forall i \in 1..n \tag{20}$$

$$w_{j,k} = 1 \Leftrightarrow a_{j,k} < n + 3, \; \forall j \in 1..m, \; \forall k \in 1..t \tag{21}$$

$$W_j = \sum_{k=1}^{t} w_{j,k}, \; \forall j \in 1..m \tag{22}$$

where

$$W_j \in 0..dayDuration, \; \forall j \in 1..m$$

Model 2: Formulation of the shift problem without the Shift constraint.

denotes that an integer overflow occurred during run time. We observe that:

- Model 2 without the Shift constraint reaches better solutions for some small instances, but building it is time-consuming and needs a huge amount of memory, until it finally becomes too large to load, mainly due to the memory requirement of (18). Even though this model appears to be more appropriate for small instances, in terms of the quality of the best solutions found, it is clearly unusable in practice for medium and large instances.

- Model 1 with Decomposition 2 of the Shift constraint and with Decomposition 1 of the StableKeysort constraint takes linear building time and memory, but the domain creation of key K_i throws an integer overflow when the instance size n becomes large; indeed, the upper bound of K_i is approximately n^{k+1}, with $k = 2$ here.

- Model 1 with Decomposition 2 of the Shift constraint and with the StableKeysort propagator of Section 3.3 also takes linear building time and memory (and is slightly better), and does not suffer from memory or integer overflows.

	Model 2, without Shift			Model 1, with Decomposition 2 of Shift					
				Decomposition 1 of StableKeysort			Propagator (Section 3.3) of StableKeysort		
n	**mem.**	**build**	**best**	**mem.**	**build**	**best**	**mem.**	**build**	**best**
25	23	0.398	3144	4	0.119	3144	4	0.112	3144
50	45	0.707	4948	5	0.124	4958	5	0.120	4958
100	102	1.672	10586	8	0.155	10693	8	0.152	10693
200	246	5.999	21386	15	0.223	21386	14	0.216	21386
400	689	30.891	42772	30	0.338	42961	28	0.317	42961
800	2334	192.923	86678	64	0.472	86867	61	0.451	86867
1600	out of memory			integer overflow			141	1.447	174112
3200	out of memory			integer overflow			373	2.882	348224

Table 1: Empirical results on the shift problem: memory consumption (**mem.**, in MB), time for creating and posting all constraints (**build**, in seconds), and the best solution found in 30 minutes (**best**).

6 Conclusion

The Sort constraints, especially their StableKeysort generalisation introduced here, significantly ease the modelling of several constraints, as illustrated with our elegant decompositions of the Shift and CumulativeSmooth constraints. However, perhaps due to the identified limitations of the original Sort constraints, only a few CP solvers implement them. We argue that Sort, or ideally the new StableKeysort constraint of this paper, should be provided in every CP solver, as that will enable the widespread use of the modelling idiom we have identified. Sort is already identified as a core constraint in [4], due to its key role for constraint reification and negation purposes. Recently, MiniZinc 2 took a step in this direction by introducing sortedness constraints with stability,[1] but without keyed tuples. Our StableKeysort constraint could be added to MiniZinc, with Decomposition 1 as default, but non-scalable, implementation.

Acknowledgements

Pierre Flener and Justin Pearson are supported by grants 2011-6133 and 2012-4908 of the Swedish Research Council (VR). Xavier Lorca and Charles Prud'homme are supported by the French common-laboratory grant TransOp involving the TASC team and Eurodecision. We thank Jean-Noël Monette and the anonymous reviewers for their valuable comments, and Irena Rusu for classifying the complexity of domain consistency for StableKeysort.

References

[1] Abderrahmane Aggoun and Nicolas Beldiceanu. Extending CHIP in order to solve complex scheduling and placement problems. *Mathematical and Computer Modelling*, 17(7):57–73, 1993.

[2] Nicolas Barnier, Pascal Brisset, and Thomas Rivière. Slot allocation with constraint programming: Models and results. In Hugh McLaurin, editor, *ATM 2001, the 4th USA/Europe Air Traffic Man-*

[1] See Arg_Sort at http://www.minizinc.org/2.0/doc-lib/doc-globals-sort.html

agement R&D Seminar, 2001. http://www.atmseminar.org/seminarContent/seminar4/papers/p_119_ITFODM.pdf.

[3] Nicolas Beldiceanu and Mats Carlsson. Revisiting the cardinality operator and introducing the cardinality-path constraint family. In Philippe Codognet, editor, *ICLP 2001*, volume 2237 of *LNCS*, pages 59–73. Springer, 2001.

[4] Nicolas Beldiceanu, Mats Carlsson, Pierre Flener, and Justin Pearson. On the reification of global constraints. *Constraints*, 18(1):1–6, January 2013.

[5] Noëlle Bleuzen-Guernalec and Alain Colmerauer. Narrowing a $2n$-block of sortings in $O(n \log n)$. In Gerd Smolka, editor, *CP 1997*, volume 1330 of *LNCS*, pages 2–16. Springer, 1997.

[6] Noëlle Bleuzen-Guernalec and Alain Colmerauer. Optimal narrowing of a block of sortings in optimal time. *Constraints*, 5(1–2):85–118, 2000.

[7] Miquel Bofill, Joan Espasa, Marc Garcia, Miquel Palahí, Josep Suy, and Mateu Villaret. Scheduling B2B meetings. In Barry OSullivan, editor, *CP 2014*, volume 8656 of *LNCS*, pages 781–796. Springer, 2014.

[8] Sylvain Bouveret and Michel Lemaître. New constraint programming approaches for the computation of leximin-optimal solutions in constraint networks. In Manuela M. Veloso, editor, *IJCAI 2007*, pages 62–67. AAAI Press, 2007.

[9] Mats Carlsson and Nicolas Beldiceanu. Revisiting the lexicographic ordering constraint. Technical Report T2002-17, Swedish Institute of Computer Science, 2002.

[10] Mats Carlsson and Per Mildner. SICStus Prolog—the first 25 years. *Theory and Practice of Logic Programming*, 12(1–2):35–66, 2012.

[11] Mats Carlsson, Greger Ottosson, and B. Carlson. An open-ended finite domain constraint solver. In H. Glaser, P. Hartel, and H. Kuchen, editors, *PLILP 1997*, volume 1292 of *LNCS*, pages 191–206. Springer, 1997. SICStus Prolog is available at http://sicstus.sics.se.

[12] Chiu Wo Choi, Warwick Harvey, Jimmy H. M. Lee, and Peter J. Stuckey. Finite domain bounds consistency revisited. In Abdul Sattar and Byeong Ho Kang, editors, *AI 2006, the 19th Australian Joint Conference on Artificial Intelligence*, volume 4304 of *LNCS*, pages 49–58. Springer, 2006.

[13] Gecode Team. Gecode: A Generic Constraint Development Environment, 2006. http://www.gecode.org.

[14] Fred Glover. Maximum matchings in convex bipartite graphs. *Naval Research Logistics Quarterly*, 14:313–316, 1967.

[15] Kurt Mehlhorn and Sven Thiel. Faster algorithms for bound-consistency of the sortedness and the alldifferent constraint. In Rina Dechter, editor, *CP 2000*, volume 1894 of *LNCS*, pages 306–319. Springer, 2000.

[16] Nicholas Nethercote, Peter J. Stuckey, Ralph Becket, Sebastian Brand, Gregory J. Duck, and Guido Tack. MiniZinc: Towards a standard CP modelling language. In Christian Bessière, editor, *CP 2007*, volume 4741 of *LNCS*, pages 529–543. Springer, 2007. The MiniZinc toolchain is available at http://www.minizinc.org.

[17] William J. Older, G. M. Swinkels, and Maarten H. van Emden. Getting to the real problem: Experience with BNR Prolog in OR. In *PAP 1995, the 3rd International Conference on the Practical Application of Prolog*, pages 465–478. Alinmead Software Ltd., 1995. http://webhome.cs.uvic.ca/~vanemden/Publications/PAP95.pdf.

[18] or-tools Team. or-tools: The Google Operations Research Suite, 2015. https://code.google.com/p/or-tools/.

[19] Patrick Pekczynski. Implementation and evaluation of advanced propagation algorithms for global constraints. Master's thesis, Universität des Saarlandes, Saarbrücken, Germany, 2006. http://www.ps.uni-saarland.de/theses/pekczynski/fopra/fopra.html.

[20] Gilles Pesant. A regular language membership constraint for finite sequences of variables. In Mark Wallace, editor, *CP 2004*, volume 3258 of *LNCS*, pages 482–495. Springer, 2004.

[21] Charles Prud'homme, Jean-Guillaume Fages, and Xavier Lorca. *Choco: A Free and Open-Source Java Library for Constraint Programming*. TASC, INRIA Rennes, LINA CNRS UMR 6241, COSLING S.A.S., France, 2014. http://www.choco-solver.org.

[22] Jean-Charles Régin. Generalized arc-consistency for global cardinality constraint. In Dan Weld and Bill Clancey, editors, *AAAI 1996*, pages 209–215. AAAI Press, 1996.

[23] Francesca Rossi, Peter van Beek, and Toby Walsh, editors. *Handbook of Constraint Programming*. Elsevier, 2006.

[24] Irena Rusu. NP-hardness of sortedness constraints. Technical Report 1506.02442, Computing Research Repository, June 2015. http://arxiv.org/abs/1506.02442.

[25] Pierre Schaus, Yves Deville, and Pierre Dupont. Bound-consistent deviation constraint. In Christian Bessière, editor, *CP 2007*, volume 4741 of *LNCS*, pages 620–634. Springer, 2007.

[26] Joachim Schimpf and Kish Shen. ECLiPSe—from LP to CLP. *Theory and Practice of Logic Programming*, 12(1–2):127–156, 2012.

[27] Sven Thiel. *Efficient Algorithms for Constraint Propagation and for Processing Tree Descriptions*. PhD thesis, Universität des Saarlandes, Saarbrücken, Germany, 2004. https://www.ps.uni-saarland.de/Publications/details/thiel2004efficient.html.

[28] Michael A. Trick. A dynamic programming approach for consistency and propagation for knapsack constraints. *Annals of Operations Research*, 118(1–4):73–84, 2003.

[29] Mariem Trojet, Fehmi H'Mida, and Pierre Lopez. Project scheduling under resource constraints: Application of the cumulative global constraint in a decision support framework. *Computers & Industrial Engineering*, 61(2):357–363, September 2011.

[30] Jianyang Zhou. A constraint program for solving the job-shop problem. In Eugene C. Freuder, editor, *CP 1996*, volume 1118 of *LNCS*, pages 510–524. Springer, 1996.

[31] Jianyang Zhou. A permutation-based approach for solving the job-shop problem. *Constraints*, 2(2):185–213, 1997.