Cayley's and Holland's Theorems for Residuated Lattices

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1 Introduction

Cayley's theorem for monoids states that every monoid can be embedded in the tranformation monoid of all self-maps on a set. Actually, the set itself, may be taken as the underlying set of the monoid. If the monoid is a group, then the maps can be taken to be permutations on the set.

On the other hand, Holland's theorem [3] states that every lattice-ordered group can be embedded into the lattice-ordered group of order-preserving permutations on a totally-ordered set (aka a chain). Recall that a *lattice-ordered group* (*ℓ-group* for short) is a structure $\boldsymbol{G} = \langle G, \vee, \wedge, \cdot, ^{-1}, 1\rangle$, where $\langle G, \cdot, ^{-1}, 1\rangle$ is group and $\langle G, \vee, \wedge\rangle$ is a lattice, such that multiplication preserves the order (equivalently, it distributes over joins and/or meets). Unlike in the case of Cayley, the chain cannot be taken to be the underlying lattice of the ℓ-group, as the latter may not be a chain. Also, the order-permutations on an non-totally-ordered lattice never form an ℓ-group. In that sense Holland's theorem is more sophisticated than Cayley's, as one needs to come up with an actual chain on which the group will be acting.

We establish similar results for idempotent semirings and residuated lattices. We also provide residuated and relational versions of these theorems.

A *(unital idempotent) semiring* is an algebra $\boldsymbol{R} = \langle R, +, \cdot, 1\rangle$, where $\langle R, +\rangle$ is a semilattice, $\langle R, \cdot, 1\rangle$ is a monoid and multiplication distributes over addition, i.e., we have $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$, for all $a, b, c \in R$. We write $\boldsymbol{R}^+$ for the semilattice $\langle R, +\rangle$.

If $\boldsymbol{L} = \langle L, \vee\rangle$ is a join-semilattice, the set $\mathrm{End}(\boldsymbol{L})$ of all join-semilattice endomorphisms on $\boldsymbol{L}$ forms an idempotent semiring $\mathbf{End}(\boldsymbol{L}) = \langle \mathrm{End}(\boldsymbol{L}), \vee, \circ, id\rangle$, where $\vee$ is computed pointwise, $\circ$ is the functional composition and id is the identity map on L.

A *residuated lattice* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$, where $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid and the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y\,.$$

Note that $\langle A, \vee, \cdot, 1\rangle$ is an idempotent semiring.

Recall that one can define groups as structures $\langle G, \cdot, \backslash, /, 1\rangle$, by the term equivalence: $x\backslash y = x^{-1}y$, $y/x = yx^{-1}$, and $x^{-1} = 1/x$. Therefore, ℓ-groups are term equivalent to special residuated lattices. In particular, if $\boldsymbol{G}$ is an ℓ-group, then $\langle G, \vee, \cdot, 1\rangle$ is an idempotent semiring.

For posets $\boldsymbol{P}$ and $\boldsymbol{Q}$, a map $f\colon P \to Q$ is said to be *residuated* if there is a map $f^\dagger\colon Q \to P$ such that for all $x \in P$ and $y \in Q$ we have

$$f(x) \leq y \quad \text{iff} \quad x \leq f^\dagger(y)\,.$$

The map $f^\dagger$ is called a *residual* of f. We denote by $\mathrm{Res}(\boldsymbol{P})$ the set of all residuated maps on $\boldsymbol{P}$. Residuated maps preserve arbitrary existing joins; actually, maps on complete lattices are residuated iff they preserve arbitrary joins.

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For a join-semilattice $\boldsymbol{L} = \langle L, \vee \rangle$, the set $\mathrm{Res}(\boldsymbol{L})$ of all residuated maps on $\boldsymbol{L}$ forms a subsemiring $\mathbf{Res}(\boldsymbol{L})$ of $\mathbf{End}(\boldsymbol{L})$, since residuated maps are closed under composition and pointwise join.

Recall that an idempotent semiring $\boldsymbol{R}$ such that $\boldsymbol{R}^+$ is a complete lattice forms a residuated lattice iff its multiplication distributes over arbitrary joins from both sides (see e.g. [2]) For a complete lattice $\boldsymbol{L}$, $\mathbf{Res}(\boldsymbol{L})$ is a complete idempotent semiring. Moreover it is easy to see that it is actually a residuated lattice.

Let $\boldsymbol{R} = \langle R, +, \cdot, 1 \rangle$ be an idempotent semiring. A *(left)* $\boldsymbol{R}$*-semimodule* $\boldsymbol{M}$ is a semilattice $\langle M, + \rangle$ together with a map $\star \colon R \times M \to M$ such that for all $r, r' \in R$ and $m, m' \in M$ the following identities hold:

- $r \star (m + m') = r \star m + r \star m'$,
- $(r + r') \star m = r \star m + r' \star m$,
- $r \star (r' \star m) = (r \cdot r') \star m$,
- $1 \star m = m$.

If $\boldsymbol{M}^+$ forms a complete lattice, then we call $\boldsymbol{M}$ a *complete* $\boldsymbol{R}$-semimodule.

Every semiring $\boldsymbol{R} = \langle R, \vee, \cdot, 1 \rangle$ gives rise to an $\boldsymbol{R}$-semimodule $\langle R, \vee \rangle$, where multiplication serves as the action. On the other hand, every join-semilattice $\boldsymbol{L} = \langle L, \vee \rangle$ can be viewed as an $\mathbf{End}(\boldsymbol{L})$-semimodule, where the action $\star \colon \mathrm{End}(\boldsymbol{L}) \times L \to L$ is defined by $f \star m = f(m)$.

2 Cayley-type representation theorems

We can swiftly observe Cayley's theorem for idempotent semirings.

Theorem 1 (Cayley's theorem for idempotent semirings)**.** *Every idempotent semiring* $\boldsymbol{R}$ *embeds into* $\mathbf{End}(\boldsymbol{R}^+)$.

Given an idempotent $\boldsymbol{R}$-semimodule $\boldsymbol{M}$, we denote by $\mathcal{I}(\boldsymbol{M})$ the semimodule $\langle \mathcal{I}(\boldsymbol{M}^+), \vee \rangle$ of all ideals of $\boldsymbol{M}^+$ where $* \colon R \times \mathcal{I}(\boldsymbol{M}^+) \to \mathcal{I}(\boldsymbol{M}^+)$ is defined by $r * I = {\downarrow}\{r \star m \mid m \in I\}$.

Theorem 2 (Residuated Cayley's theorem for idempotent semirings)**.** *Any idempotent semiring* $\boldsymbol{R}$ *is embeddable into* $\mathbf{Res}(\mathcal{I}(\boldsymbol{R}^+))$.

Note that one can identify a binary relation $R \subseteq A \times B$ with a function from A to $\mathcal{P}(B)$ mapping $a \in A$ to $R(a) = \{b \in B \mid \langle a, b \rangle \in R\}$. Furthermore, such a function lifts to a function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$, defined by $R[X] = \bigcup_{x \in X} R(x)$; note that we abuse notation by overloading the symbol R. Actually, all such lifted functions are exactly the residuated maps from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. So we identify relations from A to B with residuated maps from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are join-semilattices the above maps restrict to maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{P}(B)$. We will focus on the case where this restrictions are actually residuated maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{I}(\boldsymbol{B})$; we denote the associated set by $\mathrm{Res}(\mathcal{I}(\boldsymbol{A}), \mathcal{I}(\boldsymbol{B}))$. Note that the this set forms a join semilattice under pointwise order. We will characterize the relations that give rise to residuated maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{I}(\boldsymbol{B})$.

A relation $R \subseteq A \times B$ is called *compatible* if for all $x \in A, y \in B$:

- $R(x) \in \mathcal{I}(\boldsymbol{B})$,

- $R(x \vee y) = R(x) \vee R(y)$, where the second join is computed in $\mathcal{I}(\boldsymbol{B})$.

In other words, they can be identified with join-semilattice homomorphisms from $\boldsymbol{A}$ to $\mathcal{I}(\boldsymbol{B})$, and as such they also form a complete (since $\mathcal{I}(\boldsymbol{B})$ is complete) join semilattice that we denote by $\mathrm{REnd}(\boldsymbol{A},\boldsymbol{B})$. If $\boldsymbol{A} = \boldsymbol{B}$, we refer to R as a compatible relation on $\boldsymbol{A}$ and write $\mathrm{REnd}(\boldsymbol{A})$ for the above set.

For every compatible relation R we define the map $f_R \colon \mathcal{I}(\boldsymbol{A}) \to \mathcal{I}(\boldsymbol{B})$ by $f_R(I) = R[I]$.

Lemma 3. *For join semilattices $\boldsymbol{A}$ and $\boldsymbol{B}$, the map $\phi\colon \mathrm{REnd}(\boldsymbol{A},\boldsymbol{B}) \to \mathrm{Res}(\mathcal{I}(\boldsymbol{A}),\mathcal{I}(\boldsymbol{B}))$, where $\phi(R) = f_R$, is a join-semilattice isomorphism.*

Lemma 4. *Given a join semilattice $\boldsymbol{L} = \langle L, \vee\rangle$, $\mathbf{REnd}(\boldsymbol{L}) = \langle \mathrm{REnd}(\boldsymbol{L}), \vee, \circ, Id\rangle$ is a semiring isomorphic to $\mathbf{Res}(\mathcal{I}(\boldsymbol{L}))$, where $\circ$ is the relational composition and $Id(x) = {\downarrow}x$ (i.e., $\langle x, y\rangle \in Id$ iff $x \geq y$).*

Theorem 5 (Relational Cayley's theorem for idempotent semirings)**.** *Every idempotent semiring $\boldsymbol{R}$ is embeddable into the semiring of relations $\mathbf{REnd}(\boldsymbol{R}^+)$.*

An interior operator σ on a residuated lattice $\boldsymbol{A}$ is called a *conucleus* if $\sigma(1) = 1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$. Then the residuated lattice $\boldsymbol{A}_\sigma = \langle A_\sigma, \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x\backslash_\sigma y = \sigma(x\backslash y)$ and $x/_\sigma y = \sigma(x/y)$ (see [2]). The residuated lattice $\boldsymbol{A}_\sigma$ is called a *conuclear contraction* of $\boldsymbol{A}$.

Theorem 6 (Cayley's theorem for residuated lattices)**.** *Let $\boldsymbol{A}$ be a residuated lattice and $\boldsymbol{A}^+$ its join-semilattice reduct. Then $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\mathcal{I}(\boldsymbol{A}^+)) \cong \mathbf{REnd}(\boldsymbol{A}^+)$.*

In addition, if $\boldsymbol{A}$ is complete then $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\boldsymbol{A}^+)$.

3 Holland-type representation theorems

A semiring is called *semilinear* if it satisfies

$$u \leq h \vee ca \quad \text{and} \quad u \leq h \vee db \qquad \text{implies} \qquad u \leq h \vee cb \vee da\,. \tag{1}$$

Theorem 7 (Holland's theorem for idempotent semirings)**.** *Let $\boldsymbol{R}$ be an idempotent semiring. Then the following are equivalent:*

1. *$\boldsymbol{R}$ is semilinear.*
2. *$\boldsymbol{R}$ is embeddable into $\mathbf{End}(\boldsymbol{\Omega})$ for some chain $\boldsymbol{\Omega}$.*
3. *$\boldsymbol{R}$ is embeddable into $\mathbf{REnd}(\boldsymbol{\Omega}) \cong \mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$ for some chain $\boldsymbol{\Omega}$.*

Proof. (Rough outline) For the proof of the theorem, we first show that every semilinear module, including $\boldsymbol{R}^+$ viewed as an $\boldsymbol{R}$-module, is embeddable in the diret product of linear modules, and then obtain a single linear module by taking the ordinal sum of the individual modules. The implication from (1) to (2) then follows.

To get the embedding into a direct product of linear modules, we observe that homomorphic images into linear modules arise by taking quotients with appropriately defined *linear* ideals. The semilinearity of $\boldsymbol{R}$ turns out to guarantee that we have enough such linear ideals to separate points. Factoring by an ideal is defined in terms of an equivalence relation associated with the ideal I and is given by:

$$m \sim_I m' \qquad \text{iff} \qquad (\forall r \in R)(r \star m \in I \Leftrightarrow r \star m' \in I)\,.$$

In other words the associated partition is given by the Boolean combinations of the sets $r^{-1}[I]$, for $r \in R$. $\square$

As a corollary of the above theorem, we can easily derive the original Holland's theorem for ℓ-groups from [3]. Given a chain $\mathbf{\Omega}$, the ℓ-group of all order-preserving bijections on $\mathbf{\Omega}$ is denoted $\mathbf{Aut}(\mathbf{\Omega})$.

Corollary 8 (Holland's theorem for ℓ-groups)**.** *Every ℓ-group $\boldsymbol{G}$ is embeddable into $\mathbf{Aut}(\mathbf{\Omega})$ for some chain $\mathbf{\Omega}$.*

Theorem 9 (Holland's theorem for residuated lattices)**.** *Let $\boldsymbol{A}$ be a residuated lattice. The following are equivalent:*

1. *$\boldsymbol{A}$ satisfies $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$.*
2. *$\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{REnd}(\mathbf{\Omega})$ for a chain $\mathbf{\Omega}$.*
3. *$\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\mathbf{\Omega}')$ for a complete chain $\mathbf{\Omega}'$.*

Recall that a residuated lattice is called *prelinear* if it satisfies $1 = (x\backslash y \wedge 1) \vee (y\backslash x \wedge 1)$, *cancellative* is it satisfies $xy/y = x$ and $y\backslash yx = x$, and *semilinear* if it is a subdirect product of chains.

Corollary 10. *The following varieties of residuated lattices consist of algebras that embed into a conuclear contraction of $\mathbf{Res}(\mathbf{\Omega})$ for a complete chain $\mathbf{\Omega}$.*

1. *Prelinear residuated lattices.*
2. *ℓ-groups.*
3. *Semilinear residuated lattices.*
4. *Commutative cancellative residuated lattices.*
5. *Distributive residuated lattices where multiplication distributes over meet.*

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